Diagrams, nonabelian Hodge spaces & global Lie theory

Philip Boalch IMJ-PRG & CNRS, Paris

- new results joint with Daisuke Yamakawa (Tokyo Univ. Science)
  arxiv: 1907.11149, CRAS 2020
- or by Jean Douçot arxiv: 2107.02516
- see also short survey arxiv: 1703 for more background

Lie theory

$$\mathfrak{g} \longrightarrow G$$

$$X \longmapsto \exp(X)$$

Connection $\quad \dfrac{X}{2\pi i}\dfrac{dz}{z} \longmapsto$ monodromy

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moduli spaces: $\quad \mathcal{M}^* \longrightarrow \mathcal{M}_B$ wild character variety

(same dimension)

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moduli spaces: $\quad \mathcal{M}^* \hookrightarrow \mathcal{M}_{DR} \xrightarrow{\cong} \mathcal{M}_B$ wild character variety (same dimension)

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moduli spaces:

$$\mathcal{M}^* \hookrightarrow \mathcal{M}_{DR} \xrightarrow{\cong} \mathcal{M}_B \quad \text{wild character variety (same dimension)}$$

$$\mathcal{M}_{DR} \xrightarrow{\cong} \mathcal{M} \quad \text{wild harmonic bundles (2d self-duality)}$$

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moduli spaces:

$$\begin{tikzcd}
\mathcal{M}^* \arrow[r, hook] & \mathcal{M}_{DR} \arrow[r, "\cong"] \arrow[d, leftrightarrow, "\cong", "\text{wild nonabelian Hodge}"'] \arrow[dr, "\cong"] & \mathcal{M}_B \\
& \mathcal{M}_{Dol} \arrow[r, "\cong"] & \mathfrak{M}
\end{tikzcd}$$

$\mathcal{M}_B$: wild character variety (same dimension)

$\mathcal{M}_{Dol}$: meromorphic Higgs bundles

$\mathfrak{M}$: wild harmonic bundles (2d self-duality)

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wild character variety (same dimension)

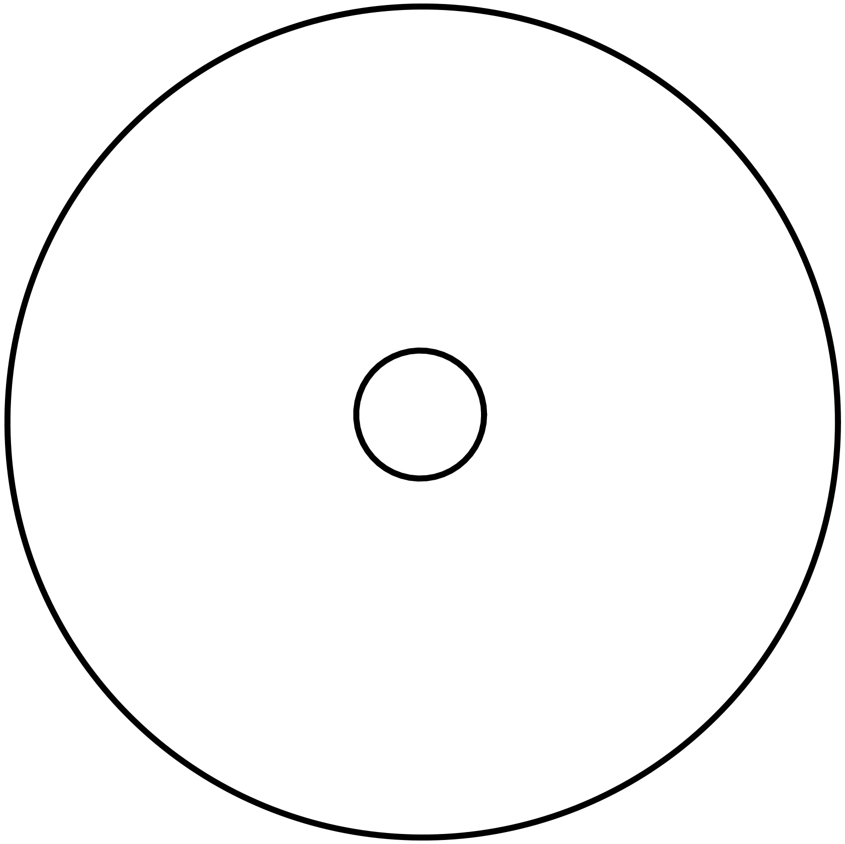
wild harmonic bundles (2d self-duality)

meromorphic Higgs bundles

"Nonabelian Hodge space"

Classify via diagrams? (e.g. sometimes $\mathcal{M}^*$ is a quiver variety)

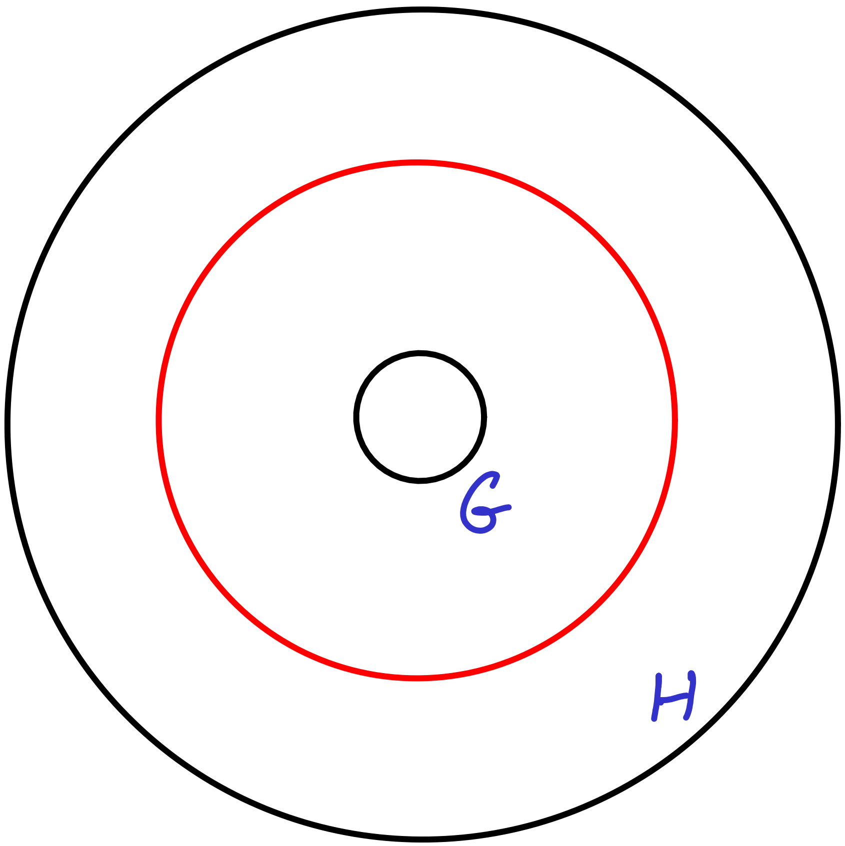
Fission spaces



Fission spaces

$V = \bigoplus_{i \in I} V_i$ $\quad$ I graded vector space

$G = GL(V) \supset H = GrAut(V) \cong \prod GL(V_i)$



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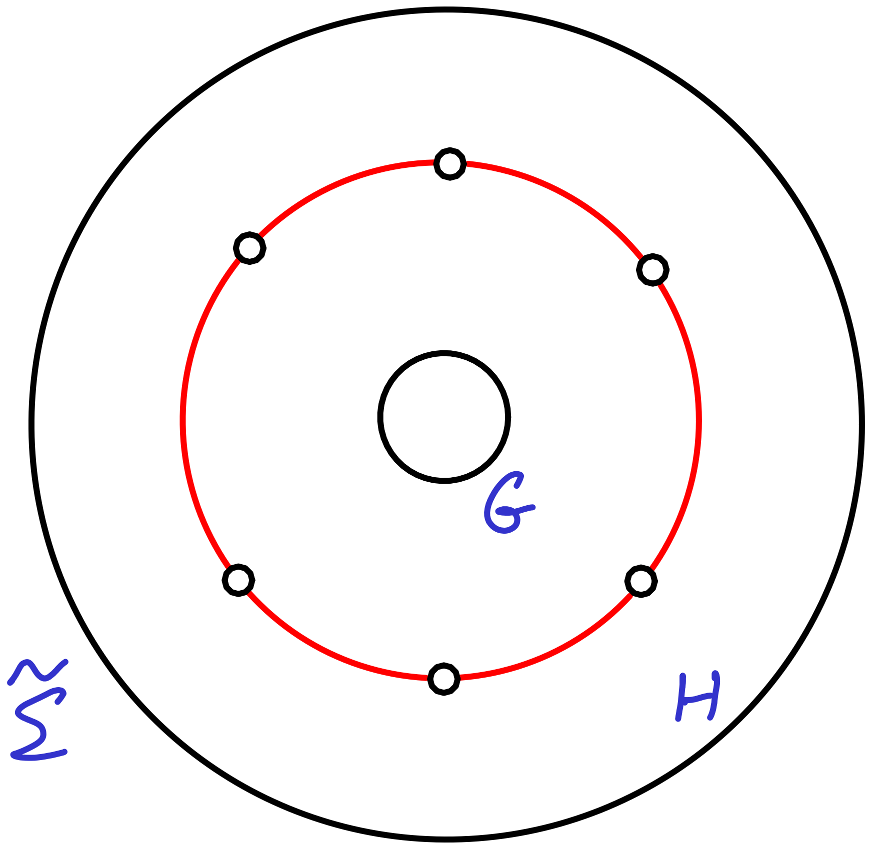

• $2k$ tangential punctures ○

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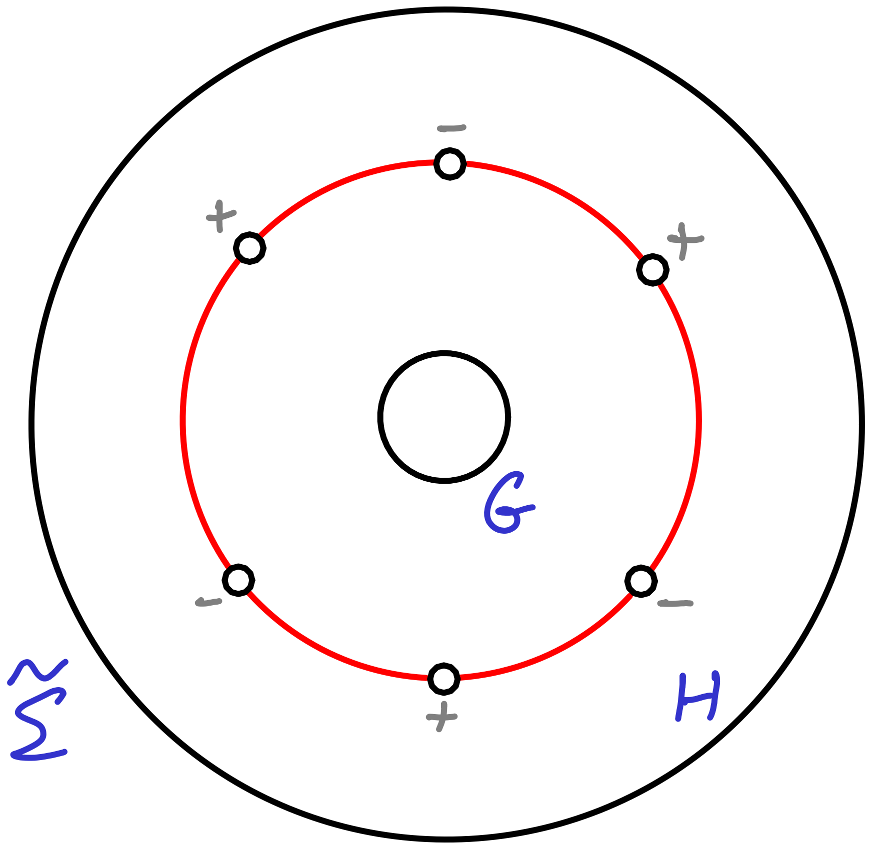

- $2k$ tangential punctures ∘
- $U_+, U_- = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ * & & 1 \end{pmatrix} \subset G$ (Stokes groups)

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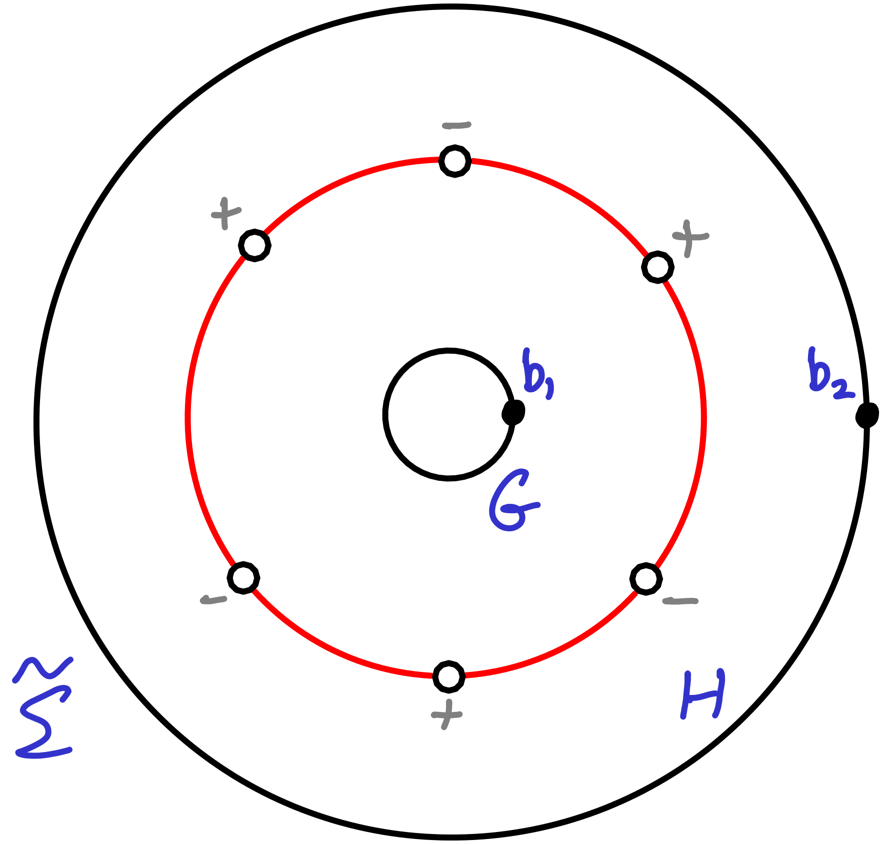

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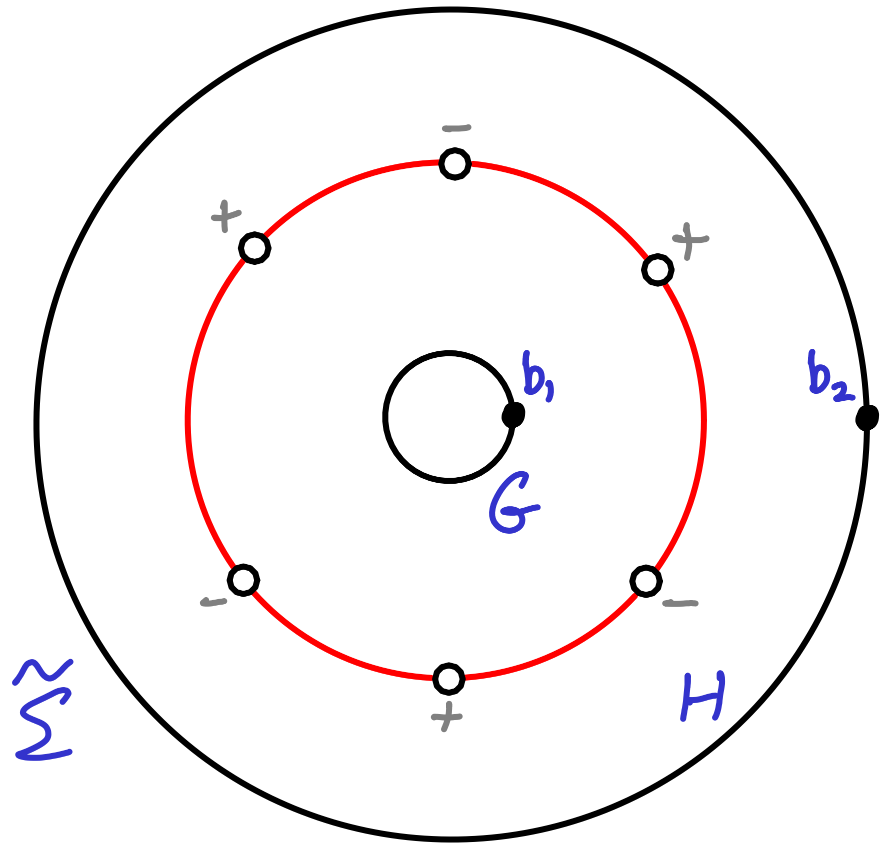

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- $\Pi = \Pi_1(\widetilde{\Sigma}, \{b_1, b_2\})$ (wild surface groupoid)
- $\mathcal{A} = {}_G\mathcal{A}^k_H = \mathrm{Hom}_{\mathbb{S}}(\Pi, G)$

$\cong G \times H \times (U_+ \times U_-)^k$

$\cong \{\text{Stokes local systems framed at } b_1, b_2\}/\text{iso.}$

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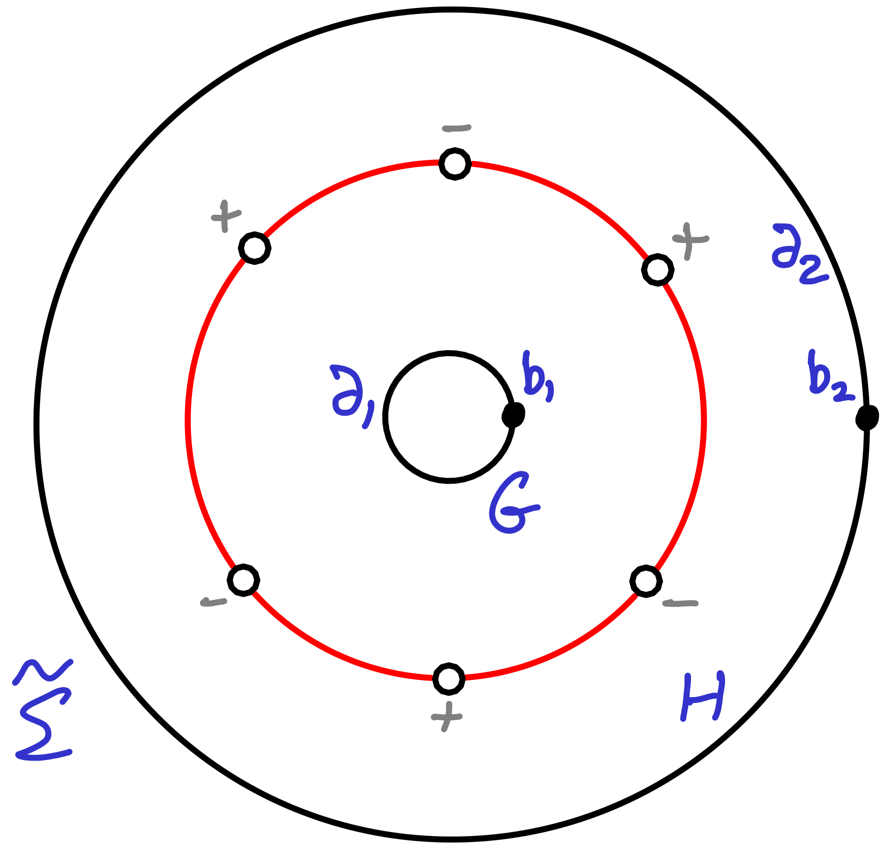

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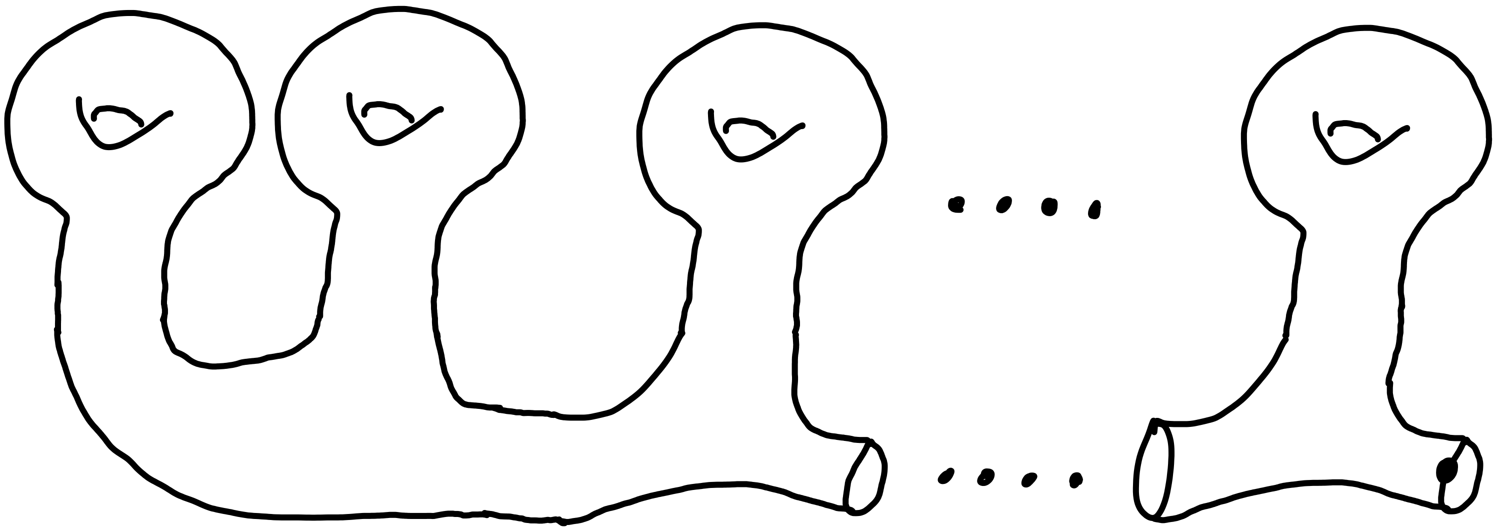
Thm $\mathcal{A}$ is a quasi-Hamiltonian $G\times H$ space with moment map $\mu : \mathcal{A} \to G \times H$, $\quad \mu(\rho) = (\rho(\partial_1), \rho(\partial_2))$

(2002 $H=T$ (any G), 2009 any H, G ($k=1$), 2011 in general)

Tame character varieties (after Alekseev–Malkin–Meinrenken 1998)



Thm. $R = \mathrm{Hom}(\pi_1(\Sigma_{g,1}), G)$ is a quasi-Hamiltonian G-space

$\cong G^{2g}$, $\mu = [A_1, B_1] \cdots [A_g, B_g] : R \to G$

$[a,b] = aba^{-1}b^{-1}$

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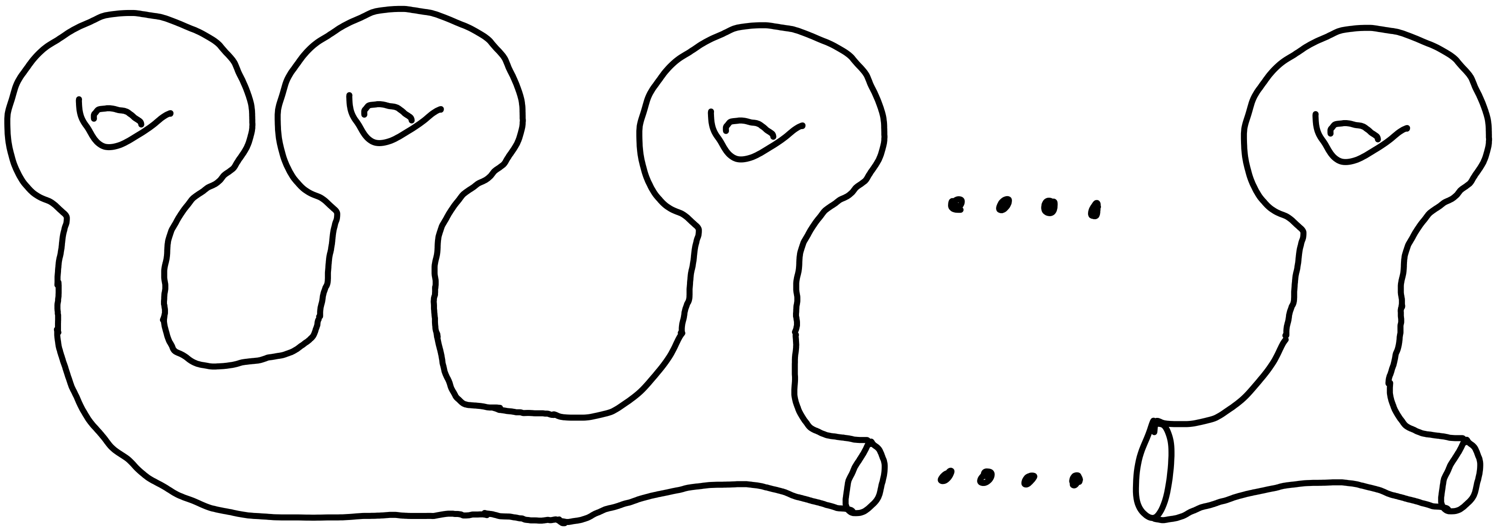

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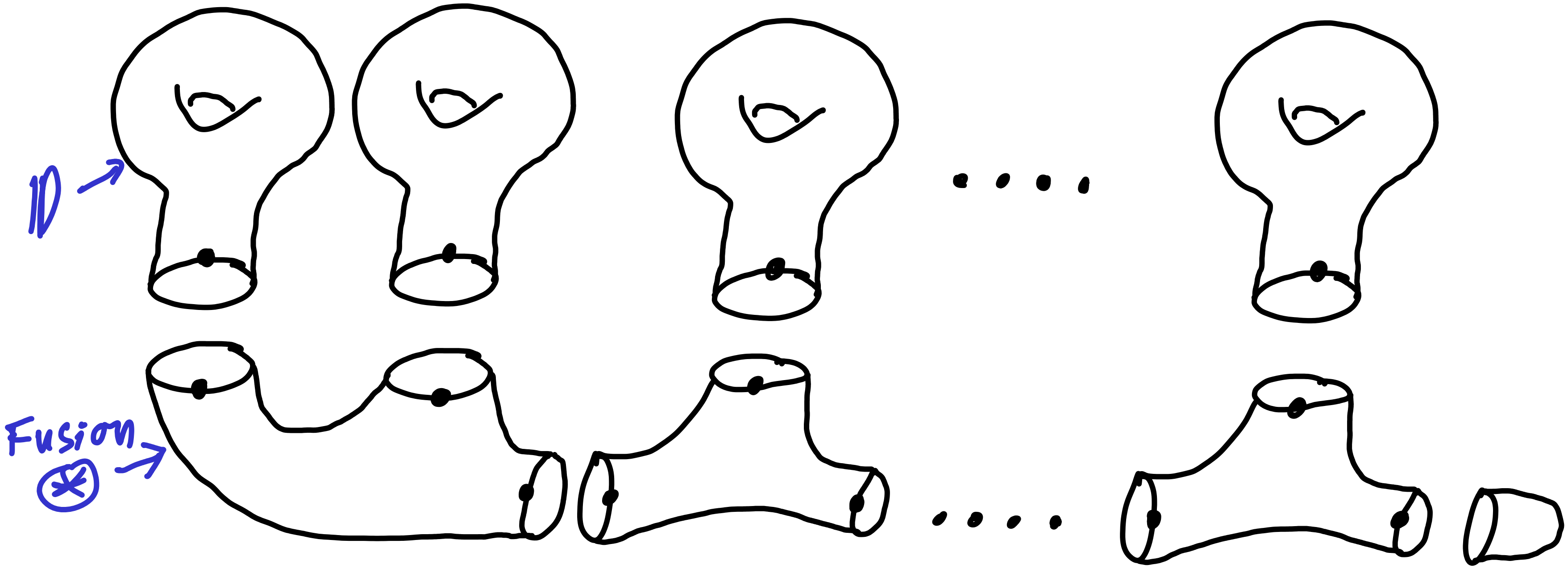
Cor.
- $\mathcal{M}_B = R/G$ is a Poisson variety
- The symplectic leaves are $\mathcal{M}_B(\mathcal{C}) = \mu^{-1}(\mathcal{C})/G$ for conjugacy classes $\mathcal{C} \subset G$

E.g. $\mathcal{M}_B(\Sigma_g) = R/\!/G = \mu^{-1}(1)/G = \{A, B \in G^{2g} \mid \prod [A_i, B_i] = 1\}/G$

Tame character varieties (after Alekseev-Malkin-Meinrenken 1998)



Thm. $\mathcal{R} = \mathrm{Hom}(\pi_1(\Sigma_{g,1}), G)$ is a quasi-Hamiltonian G-space

$$\cong G^{2g}, \quad \mu = [A_1,B_1]\cdots[A_g,B_g] : \mathcal{R} \to G$$

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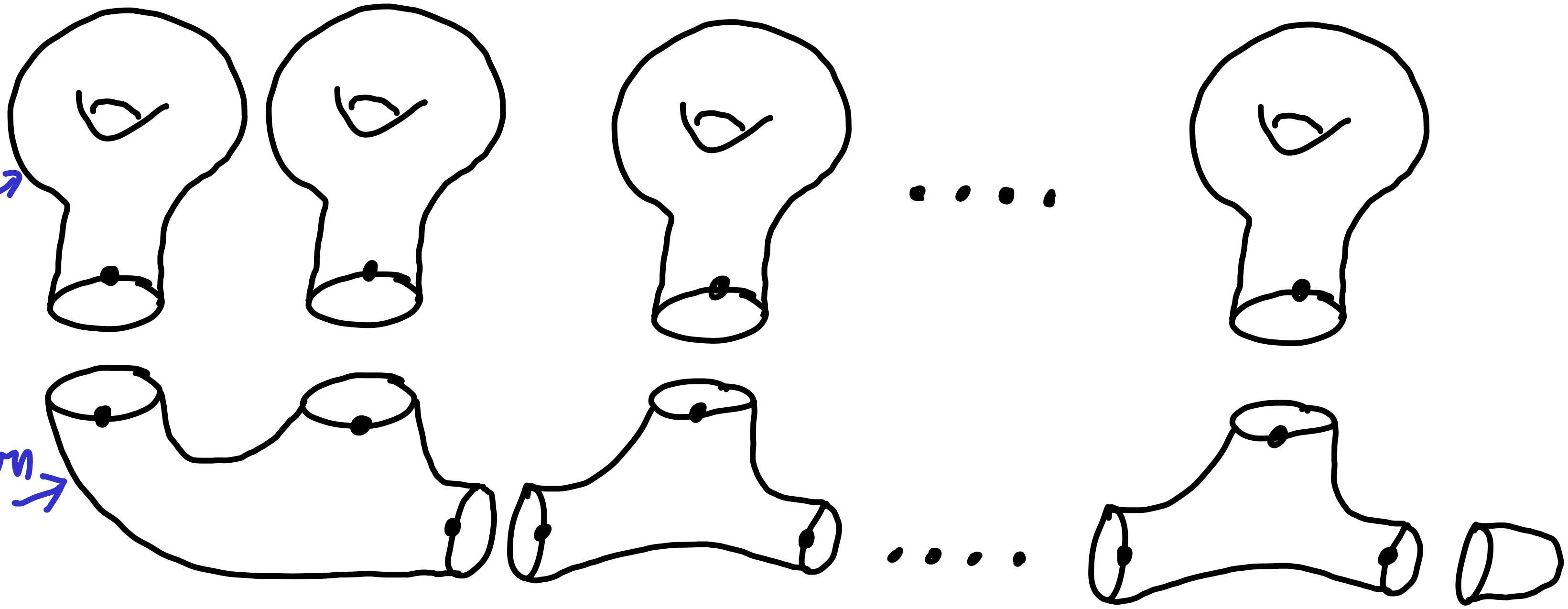
Cor.
- $\mathcal{M}_B = \mathcal{R}/G$ is a Poisson variety
- The symplectic leaves are $\mathcal{M}_B(\mathcal{C}) = \mu^{-1}(\mathcal{C})/G$ for conjugacy classes $\mathcal{C} \subset G$
- Can fuse simple pieces: $\mathcal{R} = \mathbb{D} \circledast \cdots \circledast \mathbb{D}$, $\mathbb{D} = \mathcal{R}(\Sigma_{1,1})$

Tame character varieties (after Alekseev-Malkin-Meinrenken 1998)

$\mathbb{D}$

Fusion $\circledast$

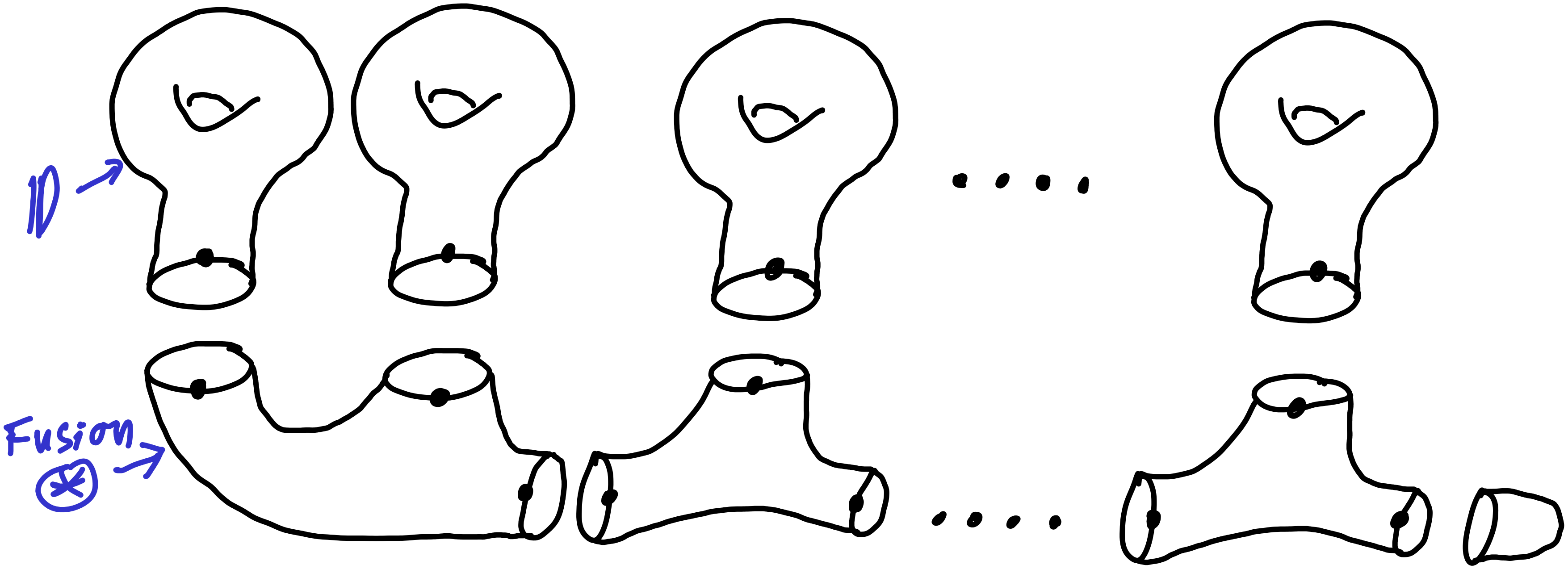


Toolbox:

- $\mathbb{D} = R(\Sigma_{1,1}) \cong G \times G$, • $\mathcal{C} \subset G$
- $D = R(\Sigma_{0,2}) = R(\boxed{\bullet\ \ \bullet}) \cong G \times G$ "double"
- $\circledast$ fusion, • $\oslash$ reduction ($/\!/G$)

$$\mathcal{M}_B(\underset{\sim}{\mathcal{C}}) = \mathbb{D} \circledast \cdots \circledast \mathbb{D} \circledast \mathcal{C}_1 \circledast \cdots \circledast \mathcal{C}_m /\!/ G$$

Tame character varieties (after Alekseev-Malkin-Meinrenken 1998)



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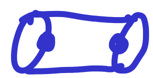
- $D = \mathcal{R}(\Sigma_{0,2}) = \mathcal{R}($  $) \cong G \times G$ "double"
- $\circledast$ fusion , • $\oslash$ reduction $(/\!/G)$

Now add fission spaces $\mathcal{A} = {}_G\mathcal{A}_H^k \quad \forall\, G, H, k$

$\Rightarrow$ lots of new algebraic symplectic/Poisson varieties

"fission varieties" $\supsetneq$ (untwisted) wild character varieties

Wild character varieties

E.g. Birkhoff 1913 wrote presentations in generic setting:

$$\left(C_1^{-1} h_1 S_{2k_1}^{(1)} \cdots S_1^{(1)} C_1\right) \cdots \left(C_m^{-1} h_m S_{2k_m}^{(m)} \cdots S_1^{(m)} C_m\right) = 1$$

(see Jimbo-Miwa-Ueno 1981 equation 2.46)

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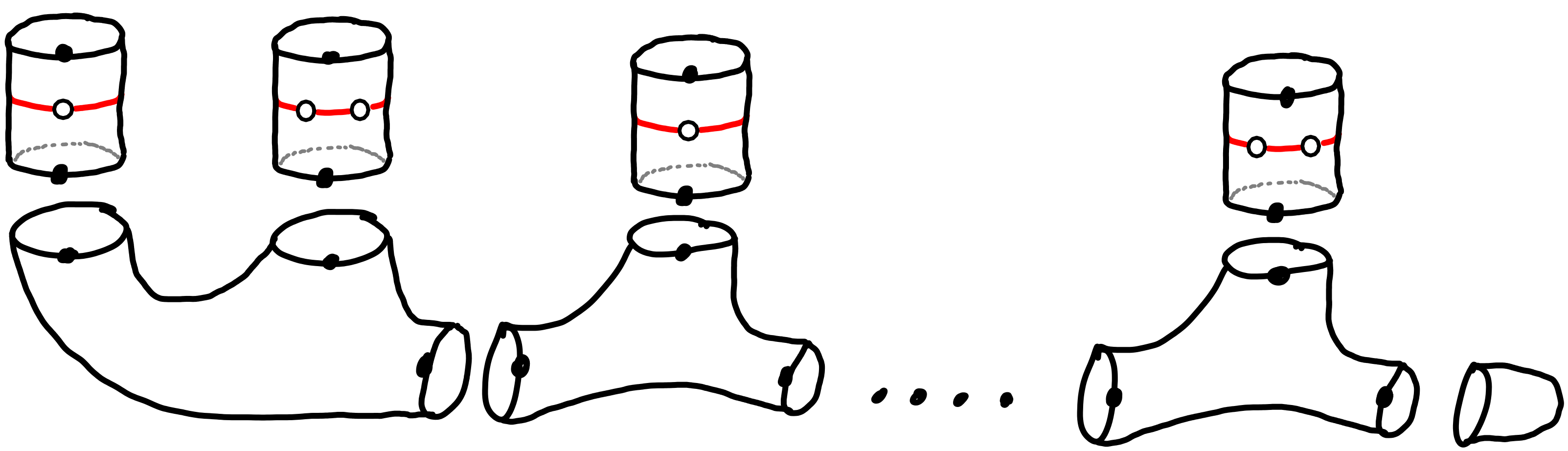
$$\mathcal{R} = {}_G\mathcal{A}_T^{k_1} \underset{G}{\circledast} {}_G\mathcal{A}_T^{k_2} \underset{G}{\circledast} \cdots \underset{G}{\circledast} {}_G\mathcal{A}_T^{k_m} \xrightarrow{\mu} T^m \times G$$



<u>Thm</u> Reductions with fixed $h_i \in T$ are symplectic

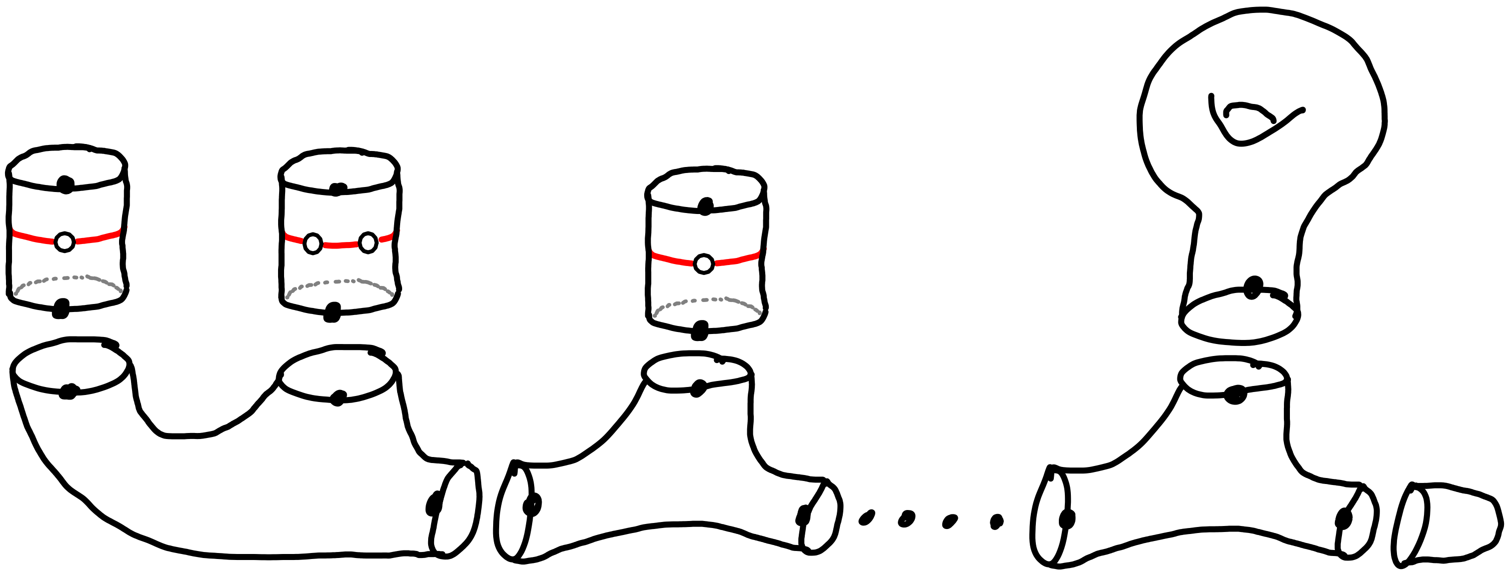
(Adv. Math. 2001 "irreg. Atiyah Bott", algebraic quasi-Hamiltonian approach 2002)

Wild character varieties

Similarly in general ($\sim$ any alg. connections on twisted G-bundles)

$$(C_1^{-1} h_1 S_{k_1}^{(1)} \cdots S_1^{(1)} C_1) \cdots (C_m^{-1} h_m S_{k_m}^{(m)} \cdots S_1^{(m)} C_m) \overrightarrow{\prod_1^g} A_i B_i A_i^{-1} B_i^{-1} = 1$$

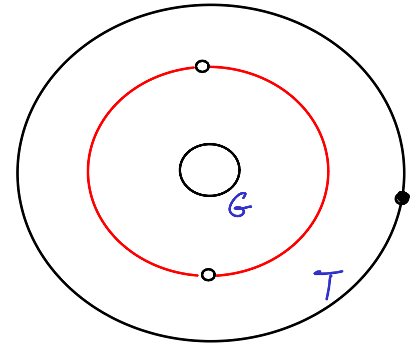
$$\mathrm{THom}_{\mathbb{S}}(\Pi, G) = \mathcal{A}_1 \circledast \cdots \circledast \mathcal{A}_m \circledast \mathbb{D}^{\circledast g} /\!/ G$$



Thm Wild character variety $\mathcal{M}_B = \mathrm{THom}_{\mathbb{S}}(\Pi, G)/\underset{\sim}{H}$ is a Poisson variety with symplectic leaves got by fixing (twisted) conjugacy classes of formal monodromy

... An. Inst Fourier '09, arXiv:1111.6228, arXiv:1512.08091 (with D. Yamakawa)

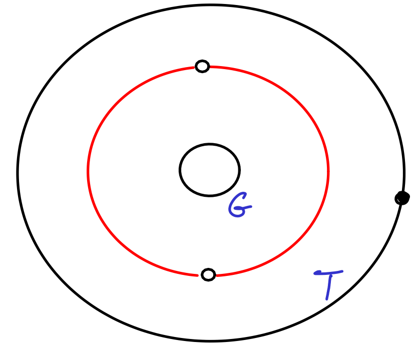
Wild character varieties



E.g. $\quad {}_G\mathcal{A}^1_T/G \cong T \times U_+ \times U_-$

is thus a nonlinear Poisson variety (with Hamiltonian T-action)

Wild character varieties



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Thm (Drinfeld/Semenov Tian Shansky, DeConcini Procesi 1993)

$U_q(\mathfrak{g})$ quantizes a Poisson variety $G^* \cong T \times U_+ \times U_-$

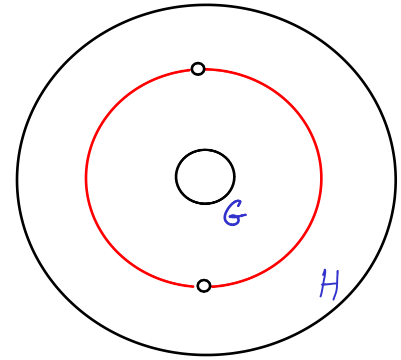
Thm (PB Invent. Math 2001)

$G^* \cong {}_G\mathcal{A}^1_T/G$ as a Poisson variety

Cor. The Drinfeld-Jimbo quantum group is modular

(comes from moduli of connections on curves)

Wild character varieties



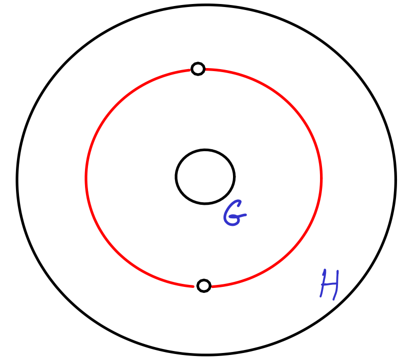
E.g. $\quad {}_G\mathcal{A}^1_H / G\times H \cong (H \times U_+ \times U_-)/H$

is an algebraic Poisson variety with symplectic leaves

$$\mathcal{M}_B(\mathcal{C}, \check{\mathcal{C}}) = \{ h, S_1, S_2 \mid h \in \check{\mathcal{C}},\ hS_1S_2 \in \mathcal{C} \}/H$$

for conjugacy classes $\check{\mathcal{C}} \subset H$, $\mathcal{C} \subset G$

Wild character varieties



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Thm (Fourier–Laplace, Malgrange 1991)

This class of varieties $\cong$ all tame genus zero character varieties

Thm – symplectic structures match too (PB arXiv 1307)

– and the hyperkähler metrics (Sz. Szabo arXiv 1407)

$\rightsquigarrow$ notion of "representations" of abstract moduli space

Plato to Painlevé

(McKay–Harnad) c.f. PB 0706.2634

Sakai's question

Exercise 3

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groups: Tetra. Octa. Icosa. $\subset$ $SO_3(\mathbb{R})$

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$\uparrow$

binary groups: $\tilde{T}$ $\tilde{O}$ $\tilde{I}$ $\subset$ $SU_2 \subset SL_2(\mathbb{C})$

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singularities: $\mathbb{C}^2/\tilde{T}$ $\mathbb{C}^2/\tilde{O}$ $\mathbb{C}^2/\tilde{I}$

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singularities:	$\mathbb{C}^2/\tilde{T}$	$\mathbb{C}^2/\tilde{O}$	$\mathbb{C}^2/\tilde{I}$		
	$\uparrow$	$\uparrow$	$\uparrow$		
resolve:	X_T	X_O	X_I		

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$$\uparrow \qquad \uparrow \qquad \uparrow$$

resolve: $\tilde{X}_T$ $\tilde{X}_O$ $\tilde{X}_I$

+deform

$$\downarrow \qquad \downarrow \qquad \downarrow$$

$\mathbb{C}^6$ $\mathbb{C}^7$ $\mathbb{C}^8$

Plato to Painlevé

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Sakai's question

Exercise 3

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 & & & & \quad \uparrow \\
\text{binary groups:} & \tilde{T} & \tilde{O} & \tilde{I} & \subset SU_2 \subset SL_2(\mathbb{C}) \\
\text{singularities:} & \mathbb{C}^2/\tilde{T} & \mathbb{C}^2/\tilde{O} & \mathbb{C}^2/\tilde{I} & \\
 & \uparrow & \uparrow & \uparrow & \\
\text{resolve:} & X_T & X_O & X_I & \\
\text{+ deform} & \tilde{\downarrow} & \tilde{\downarrow} & \tilde{\downarrow} & \\
 & \mathbb{C}^6 & \mathbb{C}^7 & \mathbb{C}^8 & \\
 & \circlearrowleft & \circlearrowleft & \circlearrowleft & \\
\text{Weyl groups:} & W(E_6) & W(E_7) & W(E_8) &
\end{array}$$

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resolve: $\uparrow$ $\uparrow$ $\uparrow$

+deform $\tilde{X}_T$ $\tilde{X}_O$ $\tilde{X}_I$

$\downarrow$ $\downarrow$ $\downarrow$

$\mathbb{C}^6$ $\mathbb{C}^7$ $\mathbb{C}^8$

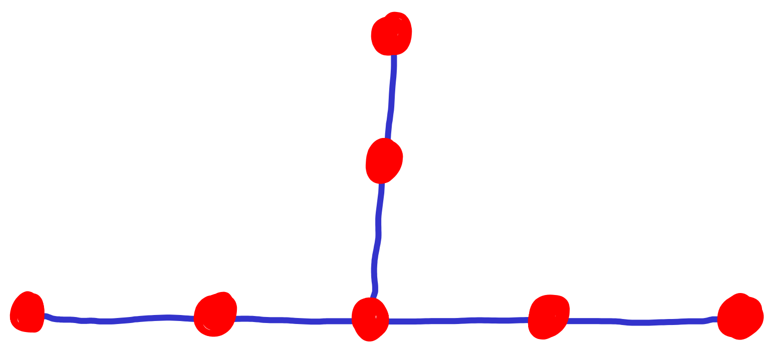
$\circlearrowleft$ $\circlearrowleft$ $\circlearrowleft$

Weyl groups: $W(E_6)$ $W(E_7)$ $W(E_8)$

Kronheimer: (1989)

- smooth fibres are complete hyperkähler 4-folds
- construct in terms of affine Dynkin graph

E.g. E_6 case (hol. symplectic approach)



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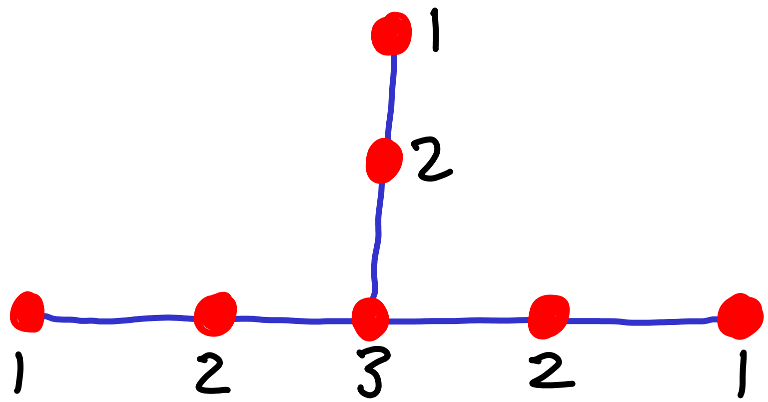
Γ

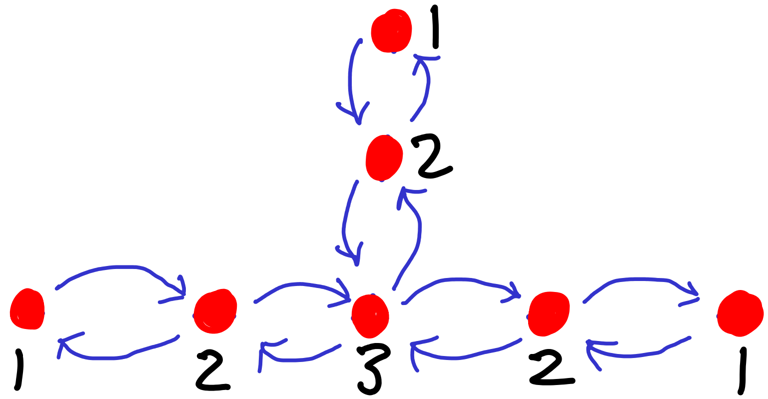


$\underset{\sim}{d}$

$$\mathbb{V} = \mathrm{Rep}(\Gamma, \mathbb{C}^{\underset{\sim}{d}})$$

E.g. E_6 case (hol. symplectic approach)

Γ



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$$\mathbb{V} = \mathrm{Rep}(\Gamma, \mathbb{C}^{\underset{\sim}{d}})$$

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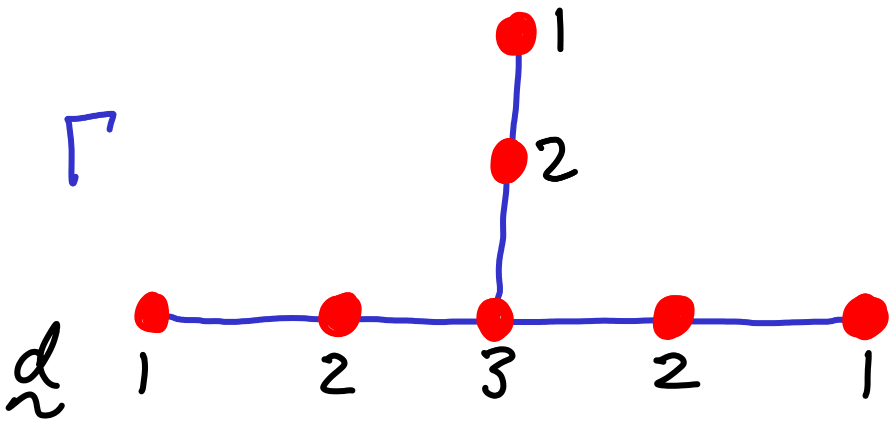

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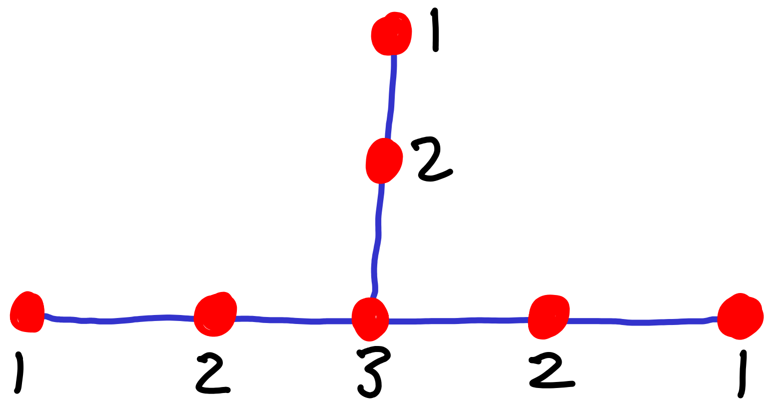

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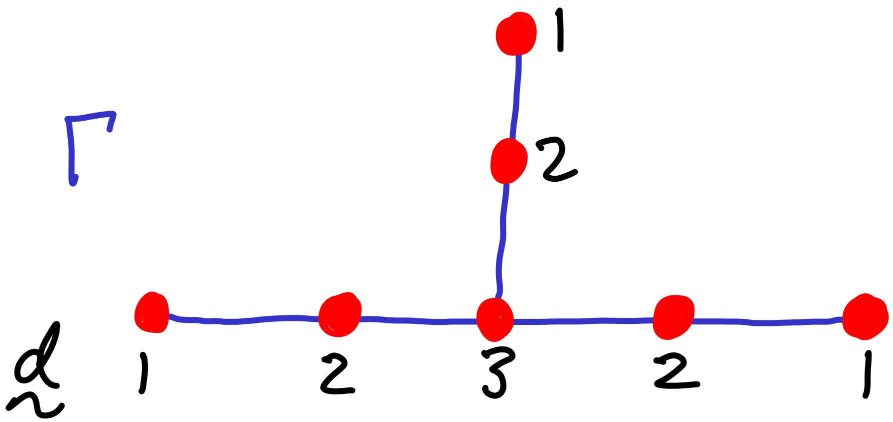

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$$\underset{\sim}{\lambda} \in \mathrm{Lie}(G)^* \cong \prod \mathrm{End}(\mathbb{C}^{d_i}) \quad \text{central}$$

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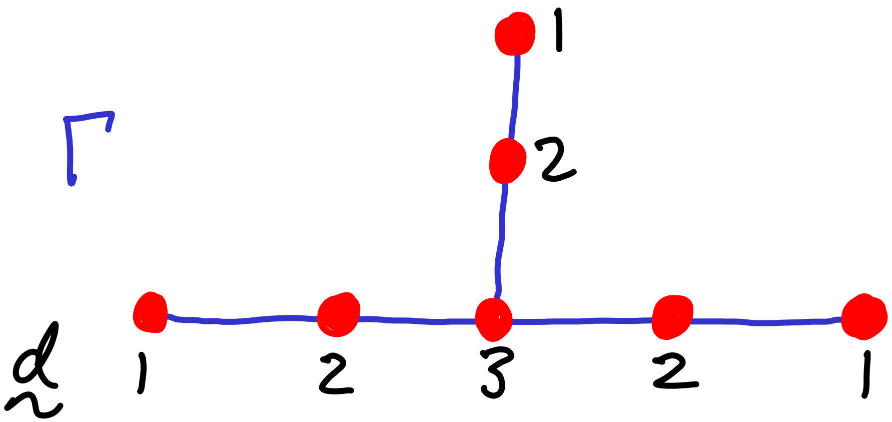

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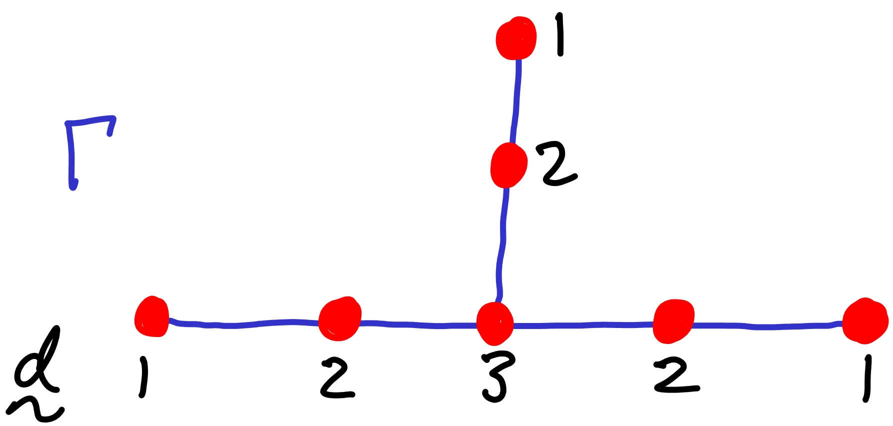

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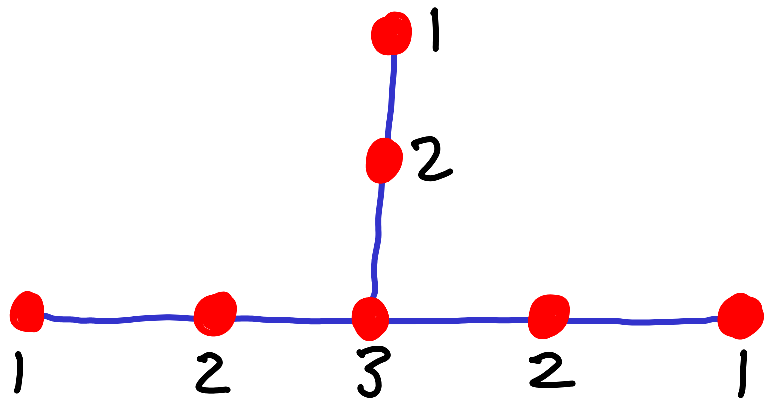

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$$\dim_{\mathbb{C}} = 6 + 6 + 6 - 2(9-1) = 2$$

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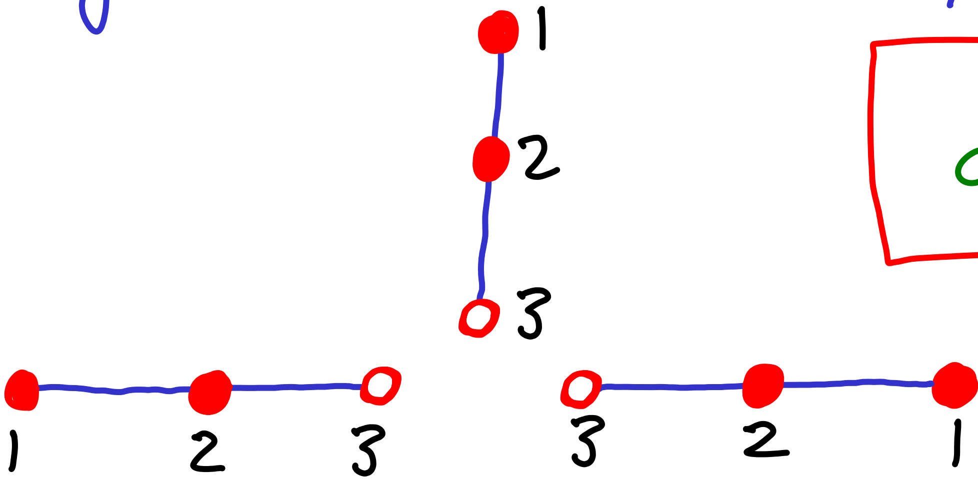

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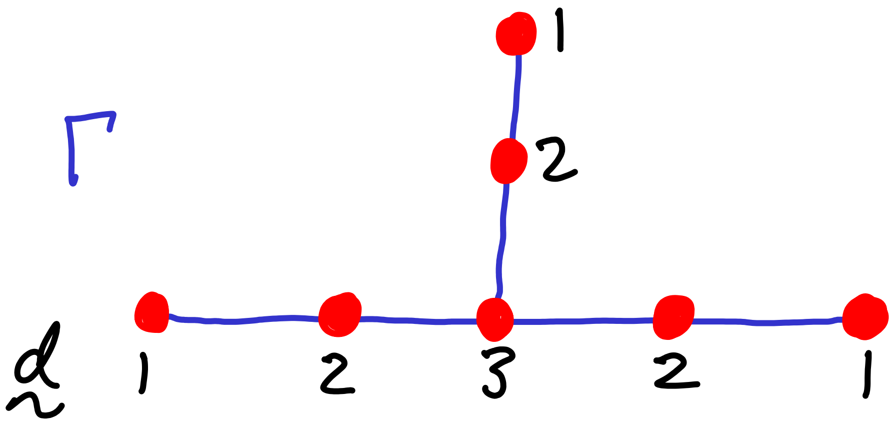

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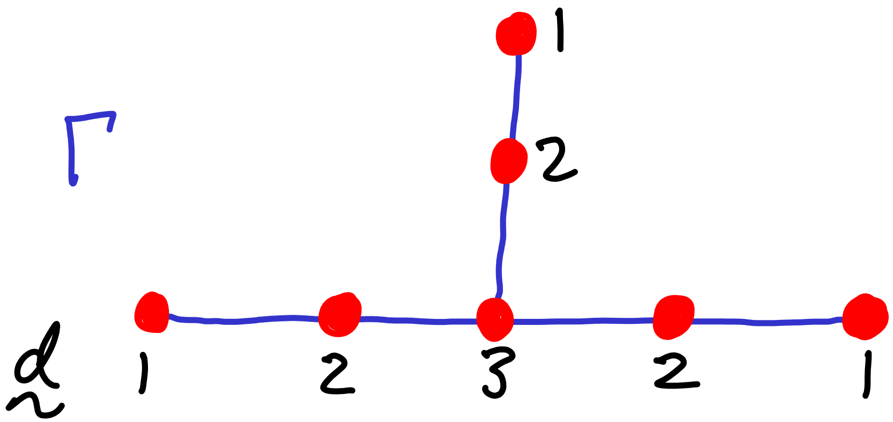

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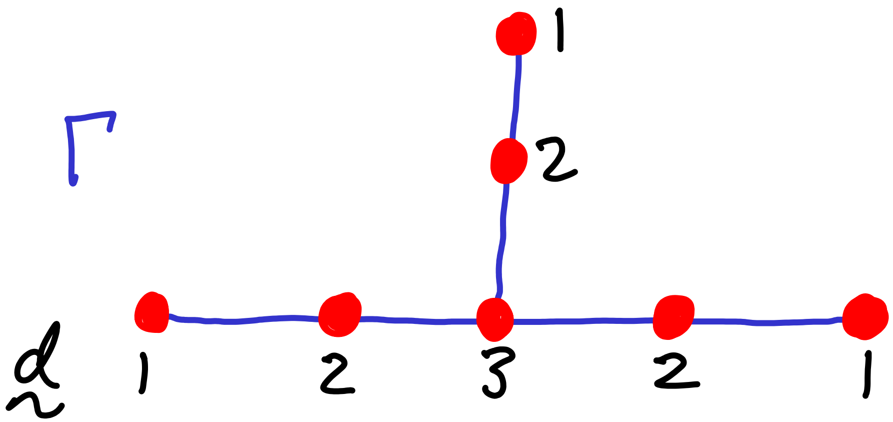

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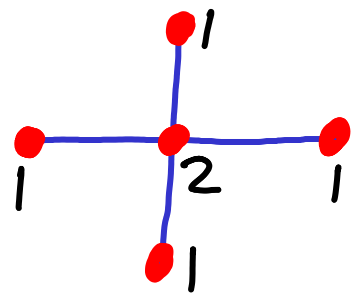
- NQV of any star-shaped Γ is modular (Kraft-Procesi, Nakajima, Crawley-Boevey)
- Get multiplicative version = character variety $\mathcal{M}_B \cong \mathcal{C}_1 \circledast \mathcal{C}_2 \circledast \mathcal{C}_3 /\!/ GL_3$

$$\mathcal{M}^* \subset \mathcal{M}_{DR} \xrightarrow{RH} \mathcal{M}_B \qquad \text{"Global Lie theory"}$$

$\exists$ one more star-shaped affine Dynkin graph:

$\hat{D}_4$



$\sim$ quaternion group $\subset SU_2$

$\{\pm 1, \pm i, \pm j, \pm k\}$

$W(D_4) \circlearrowright \mathbb{C}^4$ "constants"

Rank 2 Fuchsian systems with 4 poles $\rightsquigarrow$ cross ratio $\in \mathcal{M}_{0,4}$

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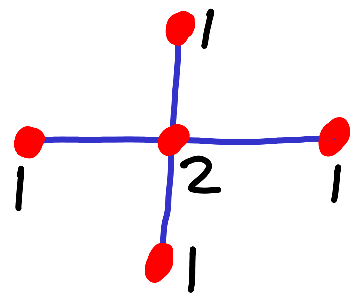

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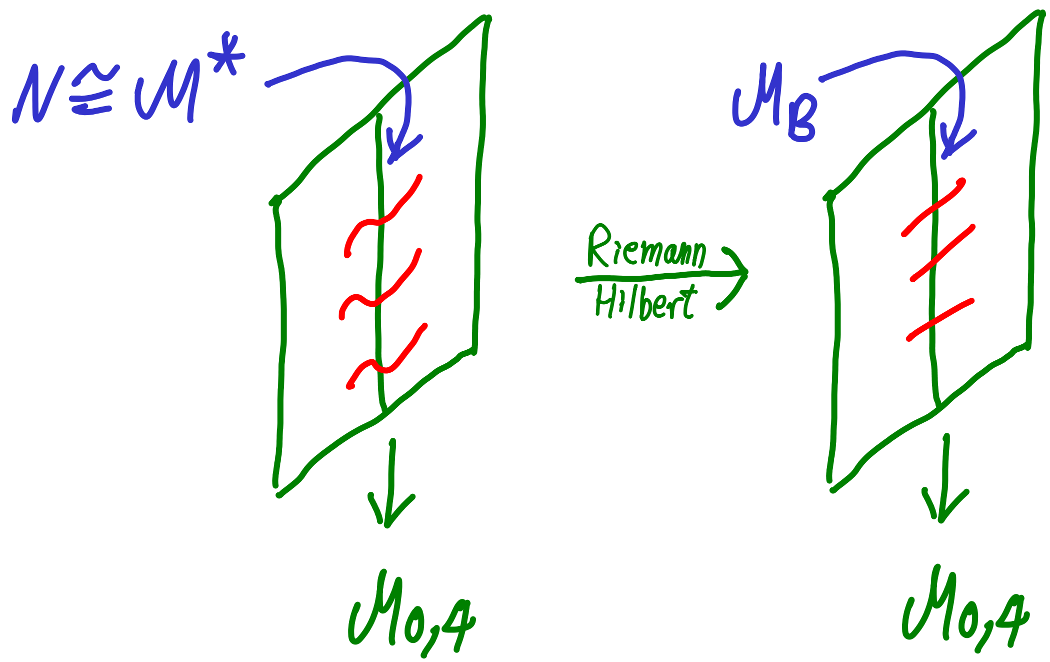

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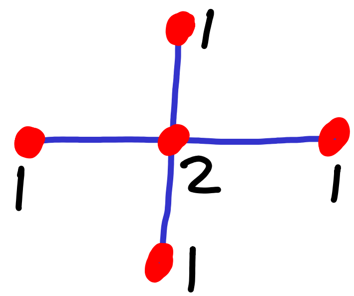

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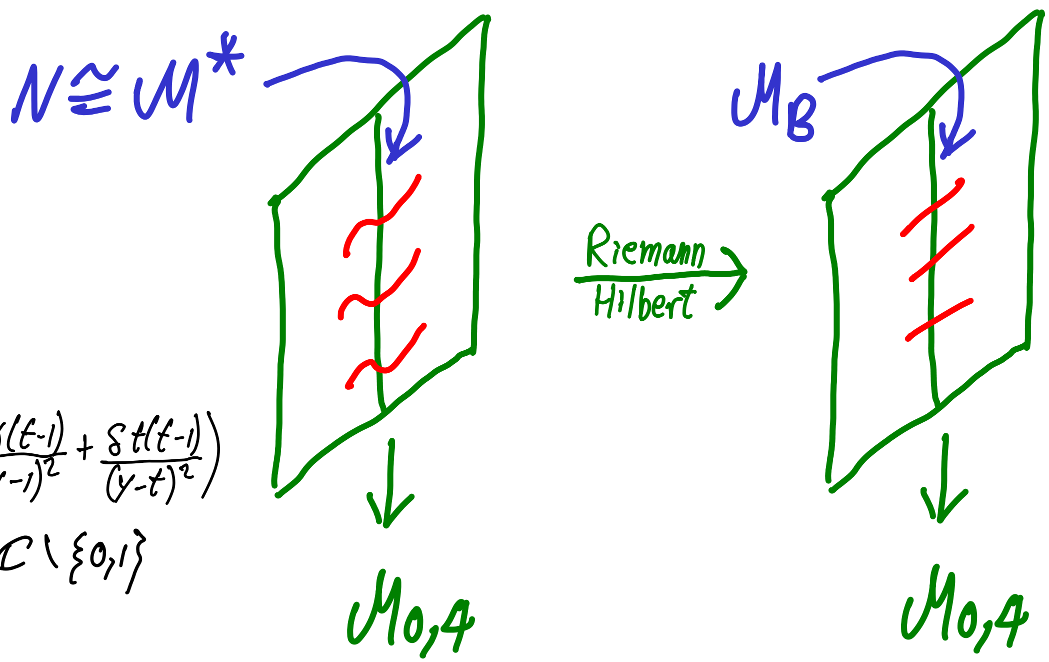

$$y'' = \left(\frac{1}{y} + \frac{1}{y-1} + \frac{1}{y-t}\right)\frac{(y')^2}{2} - \left(\frac{1}{t} + \frac{1}{t-1} + \frac{1}{y-t}\right)y' + \frac{y(y-1)(y-t)}{t^2(t-1)^2}\left(\alpha + \frac{\beta t}{y^2} + \frac{\gamma(t-1)}{(y-1)^2} + \frac{\delta t(t-1)}{(y-t)^2}\right)$$

$\alpha, \beta, \gamma, \delta \in \mathbb{C}$, $t \in \mathcal{M}_{0,4} \cong \mathbb{C} \setminus \{0,1\}$

$\mathcal{M}_B \cong$ Fricke-Klein-Vogt cubic surface

$$xyz + x^2 + y^2 + z^2 = ax + by + cz + d$$

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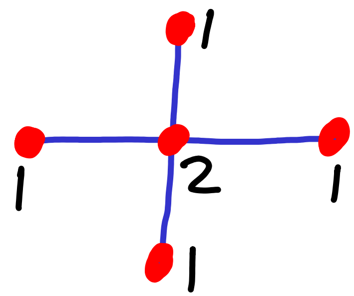

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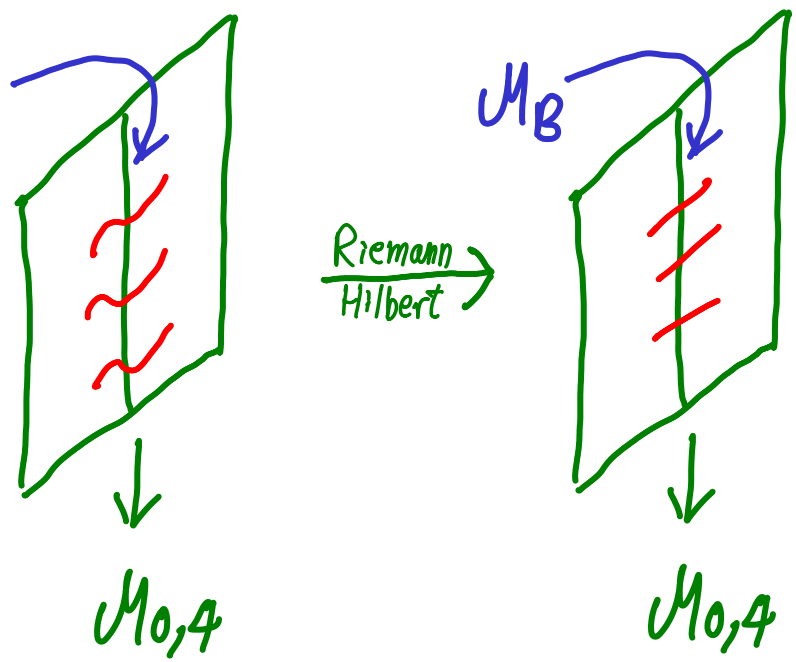
Okamoto 1987: affine Weyl group $W(\hat{D}_4) \circlearrowright \mathbb{C}^4$ relating $P\,\overline{\underline{VI}}$ equations

$N \cong \mathcal{M}^*$ $\qquad\qquad$ $\mathcal{M}_B$



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THE PAINLEVÉ EQUATIONS AND THE DYNKIN DIAGRAMS

Kazuo Okamoto

Department of Mathematics
College of Arts and Sciences
University of Tokyo
Tokyo, Japan

1 Painlevé Systems

Let δ be a differential on $\mathbf{C}(t)$, i.e.

$$\delta = f(t)\frac{d}{dt},$$

$f(t)$ being a rational function in t, and

$$\mathrm{H}(t;q,p) \in \mathbf{C}[t,q,p],$$

a polynomial in three variables (t,q,p). We consider the Hamiltonian system of ordinary differential equations:

$$\begin{aligned} \delta q &= \frac{\partial \mathrm{H}}{\partial p}, \\ \delta p &= -\frac{\partial \mathrm{H}}{\partial q}, \end{aligned} \tag{1}$$

under the assumption that H is of the second degree with respect to p. Therefore, by

P_J	1	2	3	4	5	6
δ	$\frac{d}{dt}$	$\frac{d}{dt}$	$t\frac{d}{dt}$	$\frac{d}{dt}$	$t\frac{d}{dt}$	$t(t-1)\frac{d}{dt}$
number of parameters	0	1	2	2	3	4
Affine Weyl Group	--	$\mathbf{A}_1$	$\mathbf{B}_2$	$\mathbf{A}_2$	$\mathbf{A}_3$	$\mathbf{D}_4$
Particular solutions	--	Airy	Bessel	Hermite-Weber	Confluent Hyper-geometric	Gauß' Hyper-geometric

(0706.2634) Exercise 3: works for Painlevé 5, 4, 2 too:

$$\begin{cases} \mathcal{M}^* \cong NQV(\Gamma) & (\text{ALE space of type } \hat{A}_3, \hat{A}_2, \hat{A}_1) \\ \Gamma = \text{affine Dynkin graph of Okamoto symmetry group} \end{cases}$$

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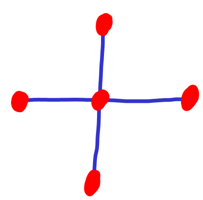
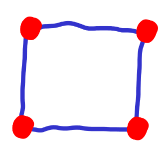
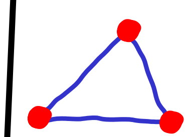
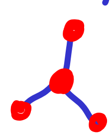
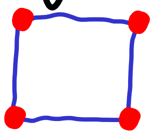
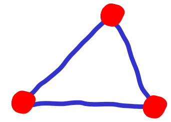
Painlevé equation	6	5	4	3	2	1
pole orders ($g=0$, $rk\,2$)	1111	211	31	$22/11\tilde{2}$	$4/1\tilde{3}$	$\tilde{4}$
# constants	4	3	2	2	1	0
Diagram				?		?
Special solutions	Gauss ${}_2F_1$ 	Kummer ${}_1F_1$?	Weber ?	Bessel-Clifford ${}_0F_1$?	Airy ?	–

Questions ① What are the higher dimensional modular quiver varieties lying over    generalising the stars?

② What about Painlevé 1 & Painlevé 3 ($\mathcal{M}^*(P_3) \ncong NQV(\Gamma)\ \forall \Gamma$) & their higher dimensional analogues?

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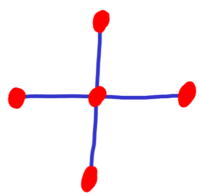
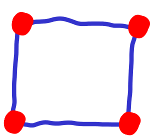
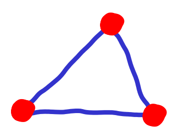
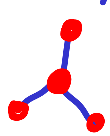
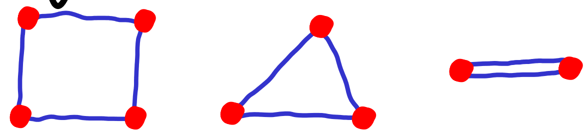
Diagram				?		?
Special solutions	Gauss $_2F_1$ 	Kummer $_1F_1$?	Weber ?	Bessel-Clifford $_0F_1$?	Airy ?	–

Questions ① What are the higher dimensional modular quiver varieties lying over  generalising the stars?

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---

③ What is the 'deeper' analogue of $\mathcal{M}_{0,4}$ in general?

$\rightsquigarrow$ moduli of wild Riemann surfaces

④ What is the 'deeper' analogue of the nonlinear local system $\mathcal{M}_B \to \mathcal{M}_{0,4}$?

$\rightsquigarrow$ local system of wild character varieties over any admissible deformation of a wild Riemann surface

[P.B. Annals of Math. 2014]

Very good connections

~ models in Biquard–B. 2004

(cf. exposition in $\begin{cases}\text{arXiv:1203.6607} \\ \text{arXiv:1703.10376}\end{cases}$)

Σ compact Riemann surface, $\underset{\sim}{a} \subset \Sigma$ finite subset

$V \to \Sigma$ holomorphic vector bundle

$\exists$ parabolic filtrations (in V_a $\forall a \in \underset{\sim}{a}$)

$\nabla : V \to V \otimes \Omega^1(* \underset{\sim}{a})$ meromorphic connection

such that ...

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such that have local bases (at each $a \in \underset{\sim}{a}$) splitting $\mathcal{F}_a$ such that:

- $\nabla = d - A$, $A = dQ + \Lambda \frac{dz}{z}$ + holomorphic terms

  $Q = \sum_1^k \frac{A_i}{z^i}$, A_i diagonal matrices (irregular type)

  $\Lambda \in \mathfrak{h}$ preserves $\mathcal{F}_a$, $\mathfrak{h} = \mathrm{Lie}(H)$, $H = C_G(Q)$

["Good" if some local cyclic pullback is very good (twisted case)]

$\rightsquigarrow \mathcal{M}_{DR}$ moduli of stable connections, $\underset{\sim}{Q}$, $Gr(\Lambda)$, parabolic weights fixed

Very good Higgs bundles

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Choices:

- $(\Sigma, \underset{\sim}{a}, \underset{\sim}{Q})$ "wild Riemann surface" (modular parameters)
- eigenvalues/orbit of Λ, parabolic weights (constants)

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Qn 1 Consider $(\mathbb{P}^1, \infty, Q)$ very good, 1 pole, wts zero

$\mathcal{M}^* \subset \mathcal{M}_{DR}$ where $V \to \mathbb{P}^1$ trivial

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Quiver modularity theorem

{ PB simply laced case + general conjecture
{ Hiroe-Yamakawa general proof

$$\mathcal{M}^*(Q, \Lambda) \cong \mathcal{N}_{Q}\mathcal{V}(\Gamma, \underset{\sim}{\xi}, \underset{\sim}{d})$$ for some Γ

Choices: • $(\Sigma, \underset{\sim}{a}, \underset{\sim}{Q})$ "wild Riemann surface" (modular parameters)

• eigenvalues/orbit of Λ, parabolic weights (constants)

Qn ① Consider $(\mathbb{P}^1, \infty, Q)$ very good, 1 pole, wts zero

$$\mathcal{M}^* \subset \mathcal{M}_{DR} \quad \text{where } V \to \mathbb{P}^1 \text{ trivial}$$

Quiver modularity theorem {PB simply laced case + general conjecture / Hiroe-Yamakawa general proof

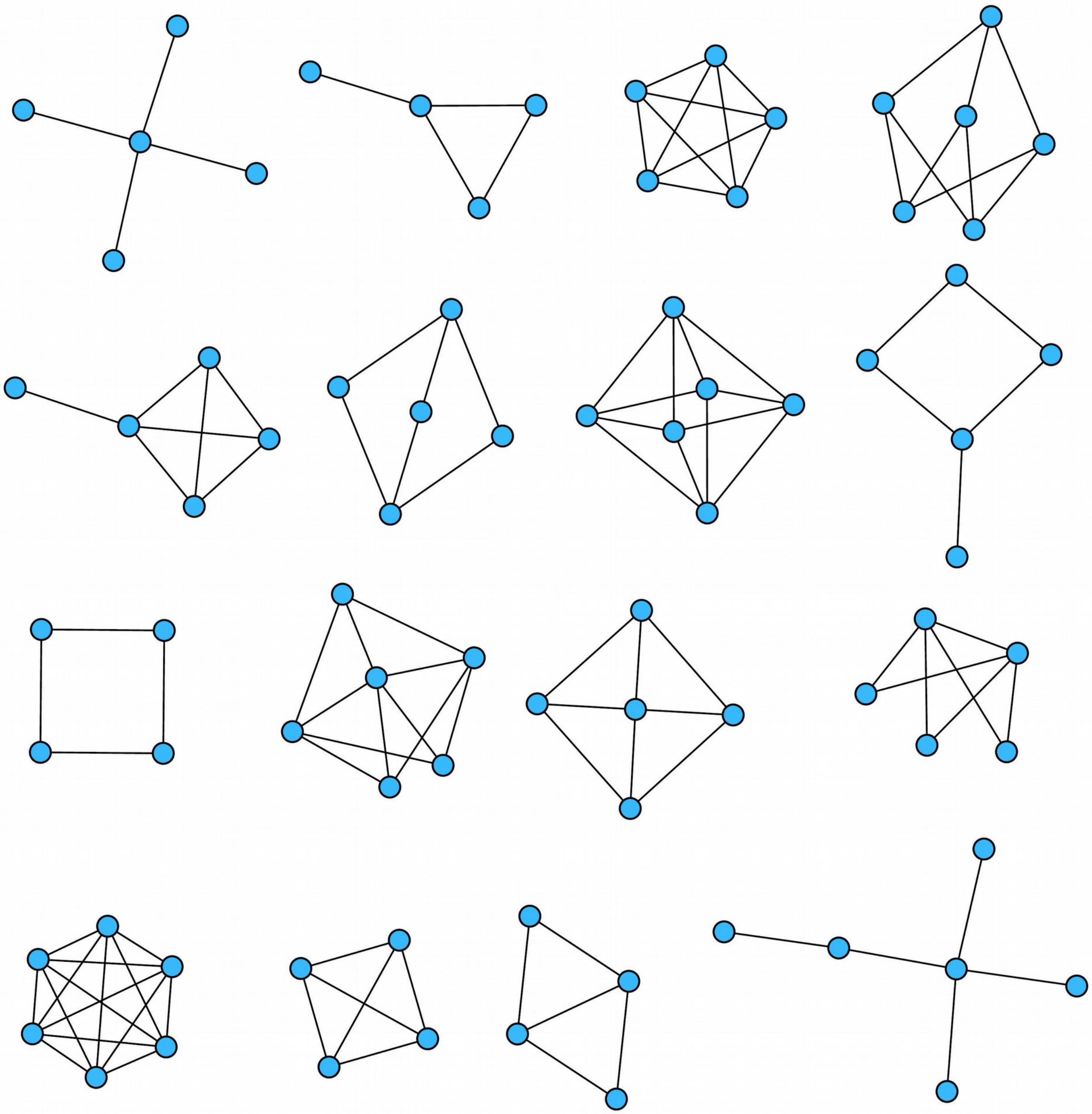
$$\mathcal{M}^*(Q, \Lambda) \cong NQV(\Gamma, \underset{\sim}{\lambda}, \underset{\sim}{d}) \quad \text{for some } \Gamma$$

"supernova graphs" (core + legs)

$$Q = \begin{pmatrix} q_1 & & \\ & \ddots & \\ & & q_n \end{pmatrix}, \quad q_i \in x\mathbb{C}[x]$$

core nodes $= \{q_i\}$, $\#\text{edges}(q_i, q_j) = \deg(q_i - q_j) - 1$

+ legs from $\Lambda \in \mathfrak{h} = \prod \mathfrak{gl}_{d_i}(\mathbb{C})$



<u>Idea</u> $\mathcal{M}^* \cong \mathcal{O} /\!\!/ G$

$$\begin{cases} dQ + \Lambda \frac{dz}{z} \in \mathcal{O} \subset \mathfrak{g}_k^* \\ G_k = GL_n(\mathbb{C}[z]/z^{k+1}) \end{cases}$$

$\cong H /\!\!/ \tilde{\mathcal{O}} /\!\!/ G$ (with $H \ni \Lambda$)

"extended orbit" $\tilde{\mathcal{O}} \circlearrowleft G \times H$

$$1 \to B_k \to G_k \xrightarrow{ev} G \to 1$$

$\cong H /\!\!/ \mathcal{O}_B$ (with $H \ni \Lambda$)

$\mathcal{O}_B \subset \mathfrak{b}_k^*$ Birkhoff orbit

B_k-coadjoint orbit of dQ

decoupling: $\boxed{\tilde{\mathcal{O}} \cong (T^*G) \times \mathcal{O}_B}$

<u>Thm</u> $\mathcal{O}_B \cong \mathbb{V}(\text{core graph})$ as a Hamiltonian H-space

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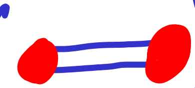
decoupling: $\boxed{\widetilde{\mathcal{O}} \cong (T^*G) \times \mathcal{O}_B}$

<u>Thm</u> $\mathcal{O}_B \cong \mathcal{W}(\text{core graph})$ as a Hamiltonian H-space

E.g. $Q = \begin{pmatrix} x^3 & \\ & -x^3 \end{pmatrix}$, $\mathcal{O}_B \cong T^*\mathbb{C}^2 = \mathcal{W}($  $)$ (Painlevé II)

$\mathcal{M}^* \cong$ Eguchi-Hanson space ($\hat{A}_1$ ALE space) $T^*\mathbb{P}^1$, $\mathcal{O} \subset sl_2(\mathbb{C})$

Qn 2

Thm (B-Yamakawa 2020)

$\exists$ uniform way to define a diagram for any meromorphic connection on $\mathbb{P}^1$ with ≤ 1 irreg. singularity

- $\dim(\mathcal{M}_B) = 2 - (\underset{\sim}{d}, \underset{\sim}{d})$ — form from Cartan matrix C of diagram
- Can have loops/edges of negative multiplicity

[Any moduli space on $\mathbb{P}^1$ has such a representation]

Qn 2

Thm (B-Yamakawa 2020, Douçot 2021)

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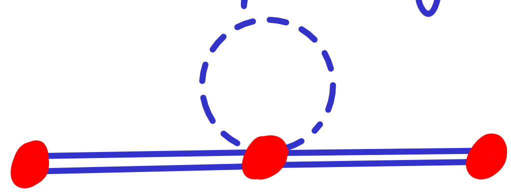
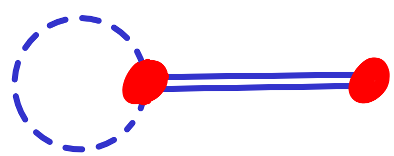
e.g Painlevé III



$$C = \begin{pmatrix} 2 & -2 & 0 \\ -2 & 4 & -2 \\ 0 & -2 & 2 \end{pmatrix}$$

$$\underset{\sim}{d} = (1, 1, 1)$$

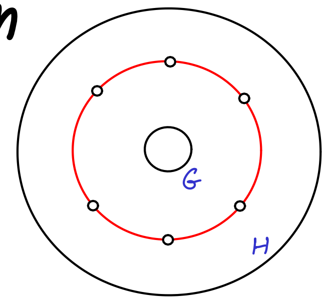
- $\sim_{\mathbb{Z}}$ Intersection form of $\mathcal{M}_{DR}$
- Weyl group $\cong W_{\text{aff}}(B_2)$ as Okamoto had
- special solutions (Bessel-Clifford)

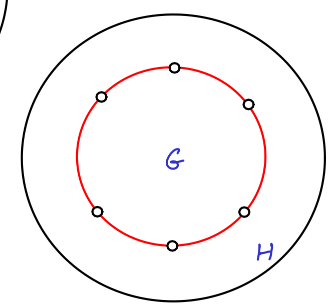


Qn 2 Idea: pass to wild character variety / & use general presentations of them

$$\mathcal{M}^* \hookrightarrow \mathcal{M}_{DR} \overset{RHB}{\cong} \mathcal{M}_B \quad \text{wild character variety}$$

	Hamiltonian	quasi-Hamiltonian
$G \times H$-spaces:	$\widetilde{\mathcal{O}}$	$\mathcal{A}$
H-spaces:	$\mathcal{O}_B$	$\mathcal{B} = \mathcal{A} /\!\!/ G$
G-spaces:	$\mathcal{O}$	$\mathcal{C} = \mathcal{A} /\!\!/_{\gamma} H$





"deeper conjugacy classes" ($\gamma = e^{2\pi i \Lambda}$)

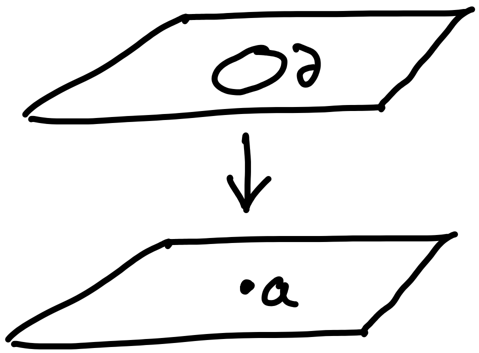
can do all this side in general twisted case (B-Y 2015)
+ looks like quiver rep. for GL_n

General choices / boundary data (twisted case) [Betti weights zero]

Fact $\exists$ covering $\mathcal{I} \to \partial$ such that:

$$\left\{ \begin{matrix} \text{connections on formal} \\ \text{punctured disk} \end{matrix} \right\} \Leftrightarrow \left\{ \begin{matrix} \mathcal{I}\text{-graded local systems} \\ \text{of vector spaces} \end{matrix} \right\}$$

[Fabry, Hukuhara, Turrittin, Levelt, Deligne]



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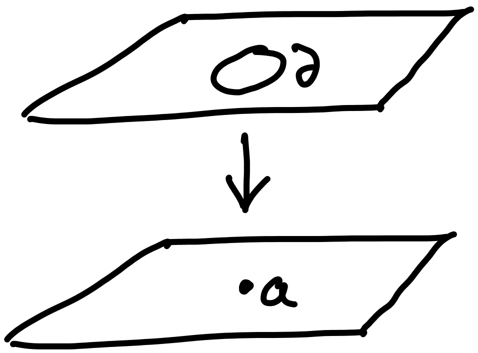

function on sector: $q = \sum_{i>0} a_i z^{-i/r}$ $(r \in \mathbb{N})$

$\Rightarrow$ circle $\langle q \rangle$ (Riemann surface / Galois orbit)

$$\begin{array}{c}\langle q \rangle \\ \downarrow \\ \partial\end{array}$$

$$\boxed{\begin{array}{cc}\tilde{I} = \bigcup \langle q \rangle & \\ \downarrow & \text{exponential local system} \\ \partial & \end{array}}$$

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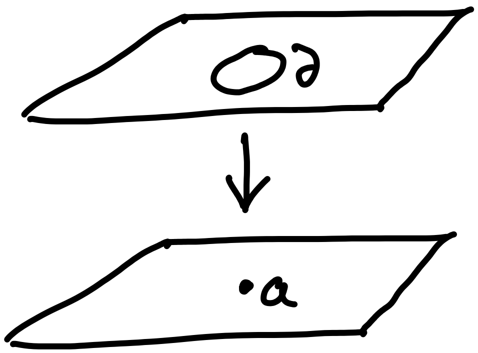

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$\mathcal{I}$-graded local system $V\to\partial$ of vector spaces

$\Longleftrightarrow$ local system $V\to\mathcal{I}$ with compact support

i.e. $V\to I$, $I\subset\mathcal{I}$ finite subcover

$\Rightarrow$ **Irregular class** $\Theta = n_1\langle q_1\rangle + \dots + n_m\langle q_m\rangle$ $n_i = \mathrm{rk}\, V|_{\langle q_i\rangle}$

$+$ **monodromy classes** $\mathcal{C}_i \subset GL_{n_i}(\mathbb{C})$

- points of maximal decay $\partial \subset \mathcal{I}$

$\partial(q) \subset \langle q \rangle$ where e^q max decay

- Irregularity: $\mathrm{Irr}(q) = \# \partial(q)$

$\mathrm{Irr}(\sum n_i \langle q_i \rangle) = \sum n_i \mathrm{Irr}(q_i)$

- Ramification: $\mathrm{Ram}(q) = \deg \pi : \langle q \rangle \to \partial$ (min r)

Choose $\Theta = \sum n_i \langle q_i \rangle$, $\mathcal{C}_i \subset GL_{n_i}(\mathbb{C})$, at $\infty \in \mathbb{P}^1$

wild Riemann surface $(\mathbb{P}^1, \infty, \Theta)$

- points of maximal decay $\partial \subset \Sigma$

$\partial(q) \subset \langle q \rangle$ where e^q max decay

- Irregularity: $\mathrm{Irr}(q) = \# \partial(q)$

$\mathrm{Irr}(\sum n_i \langle q_i \rangle) = \sum n_i \mathrm{Irr}(q_i)$

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Choose $\Theta = \sum n_i \langle q_i \rangle$, $\mathcal{C}_i \subset GL_{n_i}(\mathbb{C})$, at $\infty \in \mathbb{P}^1$

Core diagram: nodes $\sim \{\langle q_i \rangle\}$

$$\# \text{arrows } \langle q_i \rangle \to \langle q_j \rangle = B_{ij} := \begin{cases} A_{ij} - \beta_i \beta_j & i \neq j \\ A_{ii} - \beta_i^2 + 1 & i = j \end{cases}$$

$$A_{ij} := \mathrm{Irr}(\mathrm{Hom}(\langle q_i \rangle, \langle q_j \rangle)), \quad \beta_i = \mathrm{Ram}(q_i)$$

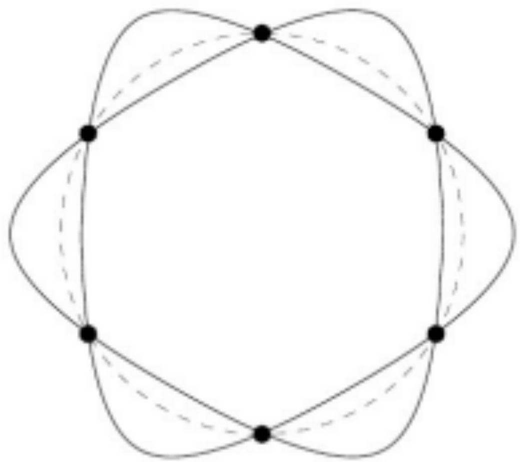
(symmetrized) Cartan matrix: $\boxed{C = 2 - B}$

Then glue on legs from classes $\mathcal{C}_i \subset GL_{n_i}(\mathbb{C})$ as before

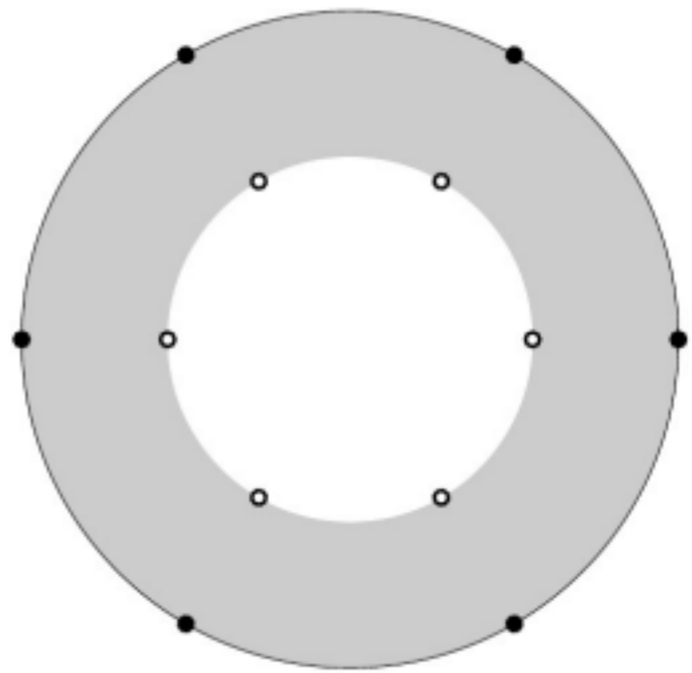
Painlevé II: $Q = \begin{pmatrix} x^3 & \\ & -x^3 \end{pmatrix}$

solutions involve e^Q

plot growth/decay of $\exp(x^3)$, $\exp(-x^3)$:



Stokes diagram with Stokes directions



Halo at ∞ with singular directions

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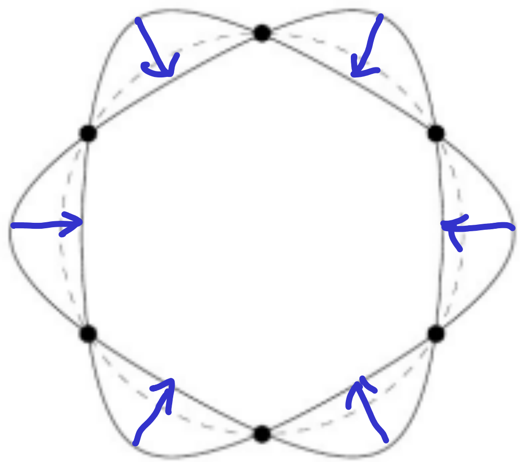

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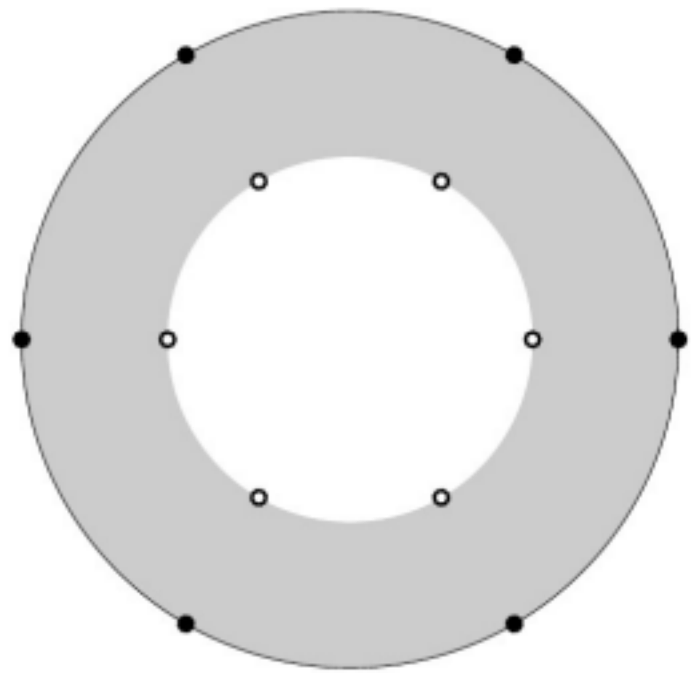

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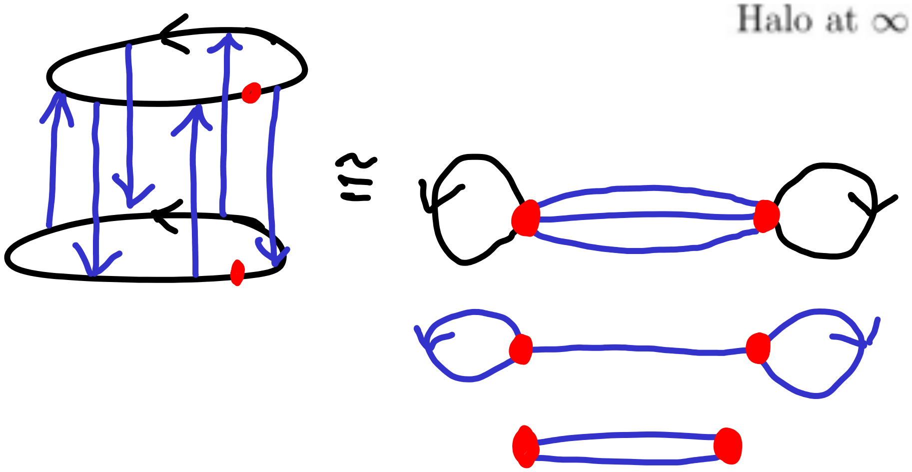


$\mu_G = 1$

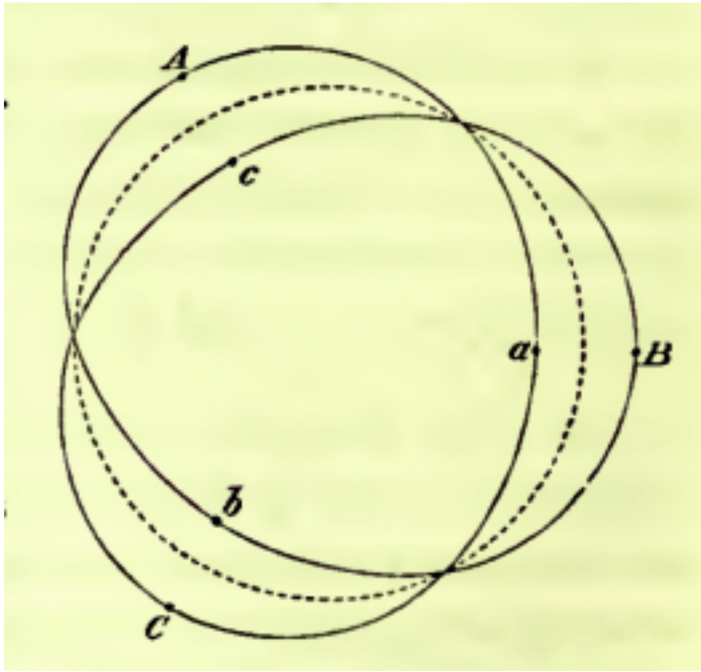
$h S_6 S_5 \cdots S_1 = 1$

2x2 matrix relation

result: $\hat{A}_1$

Airy equation (Stokes 1857)

solutions involve $\exp(x^{3/2})$



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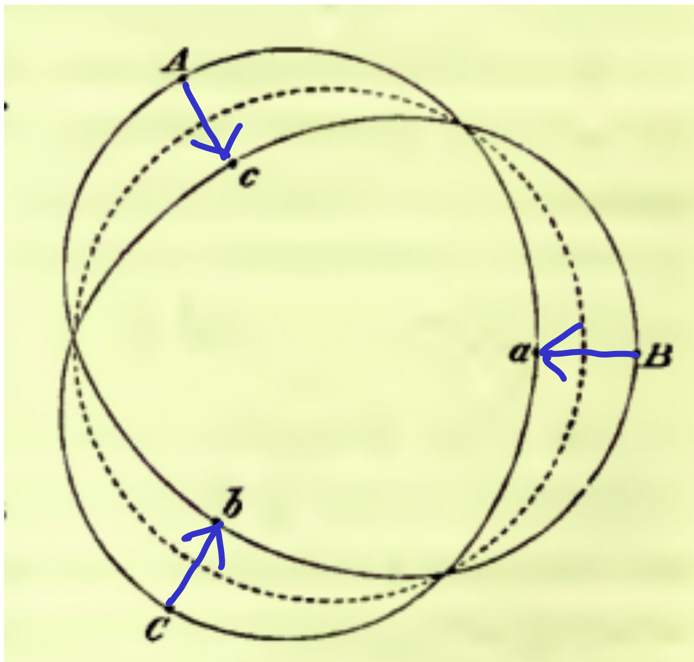
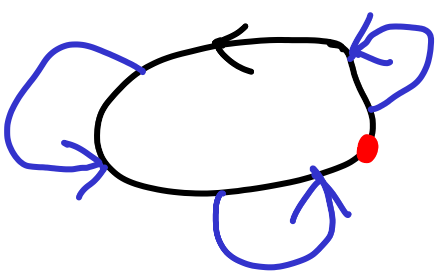
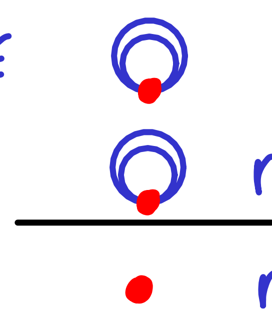


$$\mu_G = 1$$

$$\underbrace{h}_{\begin{pmatrix} 0 & * \\ 1 & 0 \end{pmatrix}} \; \underbrace{S_3}_{u_+} \; \underbrace{S_2}_{u_-} \; \underbrace{S_1}_{u_+} = 1$$



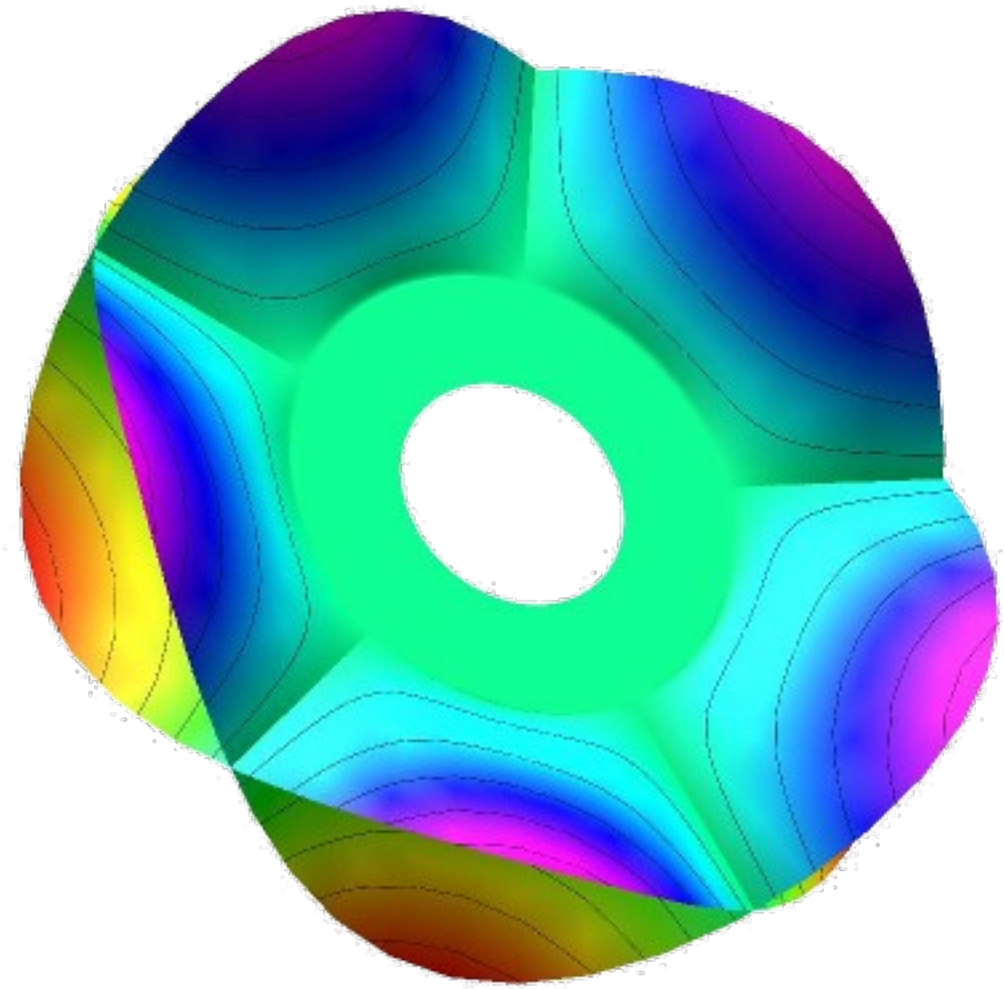
$\cong$

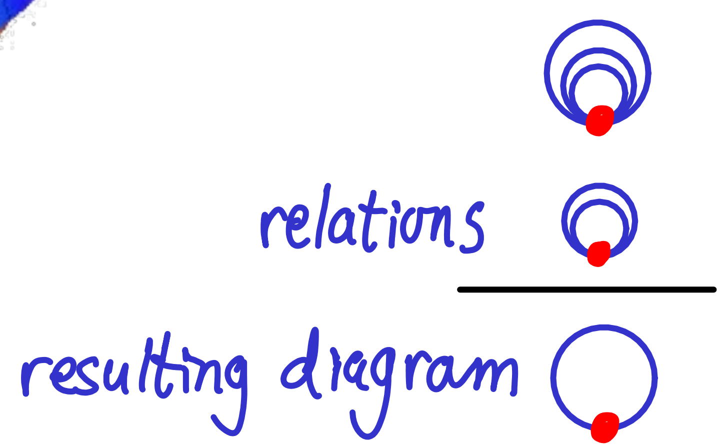


relations

resulting diagram

Painlevé 1 $\quad \exp(x^{5/2})$





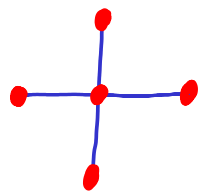
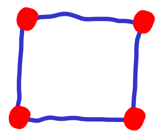
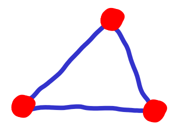
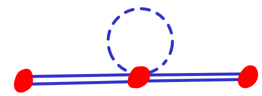
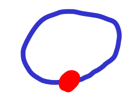
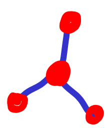
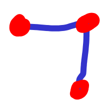
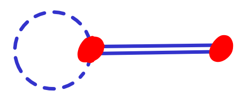
Painlevé equation	6	5	4	3	2	1
pole orders ($g=0$, $rk\,2$)	1111	211	31	22/11$\tilde{2}$	4/1$\tilde{3}$	$\tilde{4}$
# constants	4	3	2	2	1	0
Diagram						
Special Solutions	Gauss ${}_2F_1$ 	Kummer ${}_1F_1$ 	Weber 	Bessel-Clifford ${}_0F_1$ 	Airy 	–

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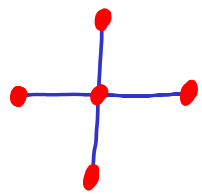
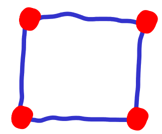
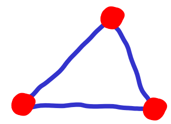
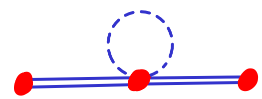
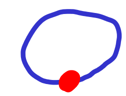
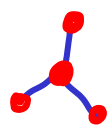
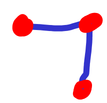
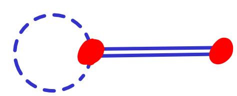
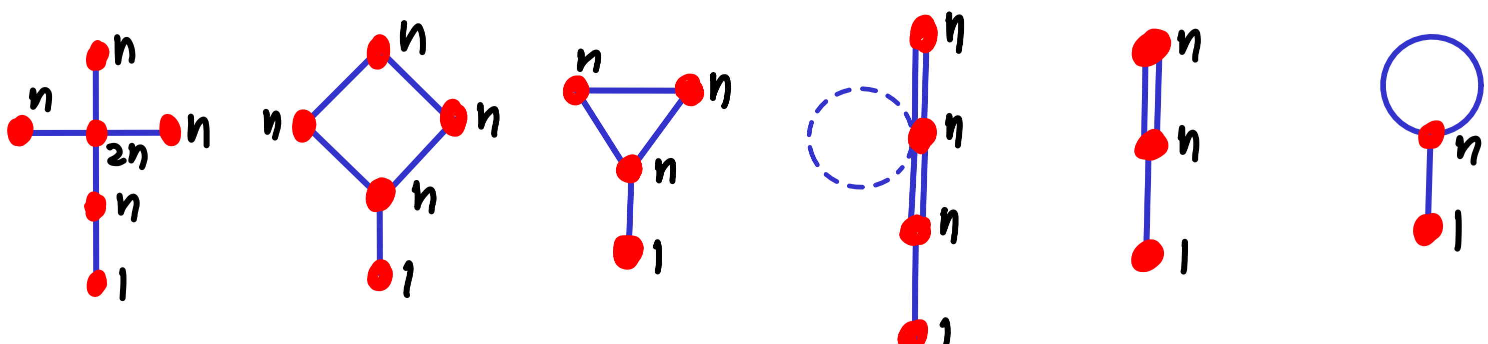
Diagram						
Special Solutions	Gauss ${}_2F_1$ 	Kummer ${}_1F_1$ 	Weber 	Bessel-Clifford ${}_0F_1$ 	Airy 	–

Higher Painlevé spaces:



dim $2n$, conjecturally $\cong \mathrm{Hilb}^n(2d\, \mathcal{M}_B)$ (known by Groechenig in tame case)